

# RESTRICTED VOLUMES AND BASE LOCI OF LINEAR SERIES

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**ABSTRACT.** We introduce and study the restricted volume of a divisor along a subvariety. Our main result is a description of the irreducible components of the augmented base locus by the vanishing of the restricted volume.

## INTRODUCTION

Let  $X$  be a smooth complex projective variety of dimension  $n$ . While it is classical that ample line bundles on  $X$  display beautiful geometric, cohomological and numerical properties, it was long believed that one couldn't hope to say much in general about the behavior of arbitrary effective divisors. However, it has recently become clear ([Na1], [Nak], [Laz], [ELMNP1]), that many aspects of the classical picture do in fact extend to arbitrary effective (or “big” divisors) provided that one works asymptotically. For example, consider the *volume* of a divisor  $D$ :

$$\mathrm{vol}_X(D) =_{\mathrm{def}} \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

When  $A$  is ample, it follows from the asymptotic Riemann-Roch formula that the volume is just the top self-intersection number of  $A$ :

$$\mathrm{vol}_X(A) = (A^n).$$

In general, one can view  $\mathrm{vol}_X(D)$  as the natural generalization to arbitrary divisors of this self-intersection number. (If  $D$  is not ample, then the actual intersection number  $(D^n)$  typically doesn't carry immediately useful geometric information. For example, already on surfaces it can happen that  $D$  moves in a large linear series while  $(D^2) \ll 0$ .) It turns out that many of the classical properties of the self-intersection number for ample divisors extend in a natural way to the volume. For instance, it was established by the second author in [Laz] that  $\mathrm{vol}_X(D)$  depends only on the numerical equivalence class of  $D$ , and that it determines a continuous function

$$\mathrm{vol}_X : N^1(X)_{\mathbf{R}} \longrightarrow \mathbf{R}$$

on the finite dimensional vector space of numerical equivalence classes of  $\mathbf{R}$ -divisors.

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*Key words and phrases.* Volume, Seshadri constant, base locus, big divisor.

*2000 Mathematics Subject Classification.* Primary 14C20; Secondary 14E15, 14E05.

The research of Ein, Lazarsfeld, Mustață and Popa was partially supported by the NSF under grants DMS 0200278, DMS 0652845, DMS 0500127, and DMS 0200150.

Now consider an irreducible subvariety  $V \subseteq X$  of dimension  $d$ . In the classical setting, when  $A$  is ample, the intersection numbers  $(A^d \cdot V)$  play an important role in many geometric questions. The goal of the present paper is to study the asymptotic analogue of this degree for an arbitrary divisor  $D$ . Specifically, the *restricted volume* of  $D$  along  $V$  is defined to be

$$\mathrm{vol}_{X|V}(D) = \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbf{C}} \mathrm{Im} \left( H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(V, \mathcal{O}_V(mD)) \right)}{m^d/d!}.$$

Thus  $\mathrm{vol}_{X|V}(D)$  measures asymptotically the number of sections of the restriction  $\mathcal{O}_V(mD)$  that can be lifted to  $X$ . For example, if  $A$  is ample then the restriction maps are eventually surjective, and hence

$$\mathrm{vol}_{X|V}(A) = \mathrm{vol}_V(A|_V) = (A^d \cdot V).$$

In general however it can happen that  $\mathrm{vol}_{X|V}(D) < \mathrm{vol}_V(D|_V)$ . The definition extends in the evident way to  $\mathbf{Q}$ -divisors. Restricted volumes seem to have first appeared in passing in Tsuji's preprint [Ts1], and they play an important role in the papers [HM], [Ta2] of Hacon–McKernan and Takayama elaborating Tsuji's work.

In order to state our results, we need to be able to discuss how  $V$  sits with respect to the base-loci of  $D$ . Recall to this end that the *stable base-locus*  $\mathbf{B}(D)$  of an integral or  $\mathbf{Q}$ -divisor  $D$  is by definition the common base-locus of the linear series  $|mD|$  for all sufficiently large and divisible  $m$ . Unfortunately, these loci behave rather unpredictably: for example, they don't depend in general only on the numerical equivalence class of  $D$ . The fourth author observed in [Na1] that one obtains a much cleaner picture if one perturbs  $D$  slightly by subtracting off a small ample class. Specifically, the *augmented base-locus* of  $D$  is defined to be

$$\mathbf{B}_+(D) =_{\mathrm{def}} \mathbf{B}(D - A)$$

for a small ample  $\mathbf{Q}$ -divisor  $A$ , this being independent of  $A$  as long as its class in  $N^1(X)_{\mathbf{R}}$  is sufficiently small. Thus  $\mathbf{B}_+(D) \supseteq \mathbf{B}(D)$ . These augmented base-loci were studied systematically in [ELMNP2], where in particular it was established that  $\mathbf{B}_+(D)$  depends only on the numerical equivalence class of  $D$ . Since the definition involves a perturbation,  $\mathbf{B}_+(\xi)$  is consequently also defined for any class  $\xi \in N^1(X)_{\mathbf{R}}$ .

Our first result involves the formal behavior of the restricted volume  $\mathrm{vol}_{X|V}(D)$ .

**Theorem A.** *Let  $V \subseteq X$  be an irreducible subvariety of dimension  $d > 0$  and let  $D$  be a  $\mathbf{Q}$ -divisor such that  $V \not\subseteq \mathbf{B}_+(D)$ . Then*

$$\mathrm{vol}_{X|V}(D) > 0,$$

*and  $\mathrm{vol}_{X|V}(D)$  depends only on the numerical equivalence class of  $D$ . Furthermore,  $\mathrm{vol}_{X|V}(D)$  varies continuously as a function of the numerical equivalence class of  $D$ , and it extends uniquely to a continuous function*

$$\mathrm{vol}_{X|V} : \mathrm{Big}^V(X)_{\mathbf{R}}^+ \longrightarrow \mathbf{R},$$

where  $\text{Big}^V(X)_{\mathbf{R}}^+$  denotes the set of all real divisor classes  $\xi$  such that  $V \not\subseteq \mathbf{B}_+(\xi)$ . This function is homogeneous of degree  $d$ , and it satisfies the log-concavity property

$$\text{vol}_{X|V}(\xi_1 + \xi_2)^{1/d} \geq \text{vol}_{X|V}(\xi_1)^{1/d} + \text{vol}_{X|V}(\xi_2)^{1/d}.$$

We also show that one can compute  $\text{vol}_{X|V}(D)$  in terms of “moving intersection numbers” of divisors with  $V$ :

**Theorem B.** *Assume as above that  $D$  is a  $\mathbf{Q}$ -divisor on  $X$ , and that  $V$  is a subvariety of dimension  $d > 0$  such that  $V \not\subseteq \mathbf{B}_+(D)$ . For every large and sufficiently divisible integer  $m$ , choose  $d$  general divisors  $E_{m,1}, \dots, E_{m,d} \in |mD|$ . Then*

$$\text{vol}_{X|V}(D) = \lim_{m \rightarrow \infty} \frac{\#(V \cap E_{m,1} \cap \dots \cap E_{m,d} \setminus \mathbf{B}(D))}{m^d}.$$

In other words,  $\text{vol}_{X|V}(D)$  computes the rate of growth of the number of intersection points away from  $\mathbf{B}(D)$  of  $d$  divisors in  $|mD|$  with  $V$ . If  $D$  is ample, this just restates the fact that  $\text{vol}_{X|V}$  is given by an intersection number. The theorem extends one of the basic properties of  $\text{vol}_X(D)$ , essentially due to Fujita; as in the case  $V = X$ , the crucial point is to show that one can approximate  $\text{vol}_{X|V}(D)$  arbitrary closely by intersection numbers with ample divisors on a modification of  $X$  (cf. [Laz] §11.4.A or [DEL]). This result has been proved independently by Demailly and Takayama [Ta2]. It also leads to an extension of the theorem of Angehrn and Siu [AS] on effective base-point freeness of adjoint bundles in terms of restricted volumes (see Theorem 2.20).

Our main result is that these restricted volumes actually govern base-loci. By way of background, suppose that  $P$  is a nef divisor on  $X$ . The fourth author proved in [Na1] that the irreducible components of  $\mathbf{B}_+(P)$  consist precisely of those subvarieties  $V$  on which  $P$  has degree zero, i.e.

$$(1) \quad \mathbf{B}_+(P) = \bigcup_{(P^{\dim V} \cdot V)=0} V,$$

where  $V$  is required to be positive dimensional. We prove the analogous result for arbitrary  $\mathbf{Q}$ -divisors  $D$ :

**Theorem C.** *If  $D$  is a  $\mathbf{Q}$ -divisor on  $X$ , then  $\mathbf{B}_+(D)$  is the union of all positive dimensional subvarieties  $V$  such that  $\text{vol}_{X|V}(D) = 0$ .*

One can extend the statement to  $\mathbf{R}$ -divisors by introducing the set  $\text{Big}^V(X)_{\mathbf{R}}$  consisting of all real divisor classes such that  $V$  is not properly contained in any irreducible component of  $\mathbf{B}_+(\xi)$ . Then  $\text{vol}_{X|V}$  determines a continuous function

$$\text{vol}_{X|V} : \text{Big}^V(X)_{\mathbf{R}} \longrightarrow \mathbf{R}$$

with the property that

$$\text{vol}_{X|V}(\xi) = 0 \iff V \text{ is an irreducible component of } \mathbf{B}_+(\xi).$$

(Note that just as it can happen for a nef divisor  $P$  that  $(P^{\dim V} \cdot V) = 0$  while  $(P^{\dim W} \cdot W) > 0$  for some  $W \subseteq V$ , so it can happen that  $\text{vol}_{X|V}(D) = 0$  but  $\text{vol}_{X|W}(D) > 0$  for

some  $W \subseteq V$ . This is why one has to focus here on irreducible components of base loci. Compare also Example 5.10.)

The proof of Theorem C is based on ideas introduced by the fourth author in [Na2], together with a result (Theorem 4.2) describing  $\text{vol}_{X|V}(D)$  in terms of separation of jets at general points of  $V$ . This allows one to lift sections of line bundles from a subvariety to  $X$  as a result of direct computation rather than vanishing of cohomology. The very rough idea of the proof is the following: starting with a lower bound for  $\text{vol}_{X|V}(D + A)$  for some ample divisor  $A$ , we deduce that there are points on  $V$  at which the line bundles  $\mathcal{O}(m(D + A))$  separate many jets, for large enough  $m$ . This allows us in turn to produce lower bounds for the dimension of spaces of sections with small vanishing order at the same points, for line bundles of the form  $\mathcal{O}(m(D - A'))$ , with  $A'$  a new ample divisor. The conclusion is that the asymptotic vanishing order of  $D - A'$  along  $V$  ( $\text{ord}_V(\|D - A'\|)$  in the notation of [ELMNP2]) can be made very small. However, as we make  $A \rightarrow 0$ , we prove that there exists a uniform constant  $\beta > 0$  such that if  $V \subseteq \mathbf{B}_+(D)$ , then  $\text{ord}_V(\|D - A'\|) > \beta \cdot \|A'\|$ , which produces a contradiction. The actual proof is quite technical and occupies most of §5.

In the last section we make the connection between our results and those of [Na2], describing the augmented base locus in terms of another asymptotic invariant, the *moving Seshadri constant*. This invariant was introduced in [Na2] as a generalization to arbitrary big divisors of the usual notion of Seshadri constant for big and nef divisors (cf. [Laz] §5.1). We describe the relationship between moving Seshadri constants and restricted volumes, and as a consequence of the results above we obtain a slight strengthening of the main result in [Na2]: *the moving Seshadri constant varies as a continuous function on  $N^1(X)_{\mathbf{R}}$ , and given an arbitrary  $\mathbf{R}$ -divisor  $D$ ,  $\mathbf{B}_+(D)$  is the set of points at which the moving Seshadri constant of  $D$  is zero.*

The results in this paper are part of a more general program of using asymptotic invariants of divisors in order to get information about the geometry of linear series, base loci, and cones of divisors on a projective variety. Invariants of a different flavor were used in [ELMNP2] in order to describe a lower approximation of the stable base locus of a divisor, called the *restricted base locus* (or *non-nef locus*, cf. [Bo1], [BDPP], see also [Deb]). The reader can also find there a thorough discussion of the connections between these various asymptotic base-locus-type constructions. Finally, we refer to [ELMNP1] for an overview of the basic ideas revolving around asymptotic invariants of line bundles.

## 1. THE AUGMENTED BASE LOCUS

We start by fixing some notation. Let  $X$  be a smooth complex projective variety of dimension  $n$ . An integral divisor  $D$  on  $X$  is an element of the group  $\text{Div}(X)$  of Cartier divisors. The corresponding linear series is denoted by  $|D|$  and its base locus by  $\text{Bs}(|D|)$ . As usual we can speak about  $\mathbf{Q}$ - or  $\mathbf{R}$ -divisors. A  $\mathbf{Q}$ - or  $\mathbf{R}$ -divisor  $D$  is *effective* if it is a non-negative linear combination of effective integral divisors with  $\mathbf{Q}$ - or  $\mathbf{R}$ -coefficients. If  $D$  is effective, we denote by  $\text{Supp}(D)$  the union of the irreducible components which appear in the associated Weil divisor. We often use the same notation for an integral divisor and

for the corresponding line bundle. Numerical equivalence between  $\mathbf{Q}$ - or  $\mathbf{R}$ -divisors will be denoted by  $\equiv$ . We denote by  $N^1(X)_{\mathbf{Q}}$  and  $N^1(X)_{\mathbf{R}}$  the finite dimensional  $\mathbf{Q}$ - and  $\mathbf{R}$ -vector spaces of numerical equivalence classes. One has  $N^1(X)_{\mathbf{R}} = N^1(X)_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ , and we fix compatible norms  $\|\cdot\|$  on these two spaces. Given a divisor  $D$ , we write  $\|D\|$  for the norm of the class of  $D$ . A  $\mathbf{Q}$ -divisor is *big* if for  $m$  divisible enough, the linear series  $|mD|$  defines a birational map onto its image. One can show that  $D$  is big if and only if  $D \equiv A + E$ , where  $A$  is ample and  $E$  is effective. This can be taken as definition in the case of an  $\mathbf{R}$ -divisor (see [Laz, Section 2.2] for the basic properties of big divisors). The *big cone* is the open convex cone in  $N^1(X)_{\mathbf{R}}$  consisting of big  $\mathbf{R}$ -divisor classes.

We recall from [ELMNP2] the definition of the augmented base locus of a divisor. Suppose first that  $D$  is a  $\mathbf{Q}$ -divisor on  $X$ . The *stable base locus* of  $D$  is

$$\mathbf{B}(D) := \bigcap_m \text{Bs}(|mD|)_{\text{red}},$$

where the intersection is over all  $m$  such that  $mD$  is an integral divisor. It is easy to see that if  $p$  is divisible enough, then  $\mathbf{B}(D) = \text{Bs}(|pD|)_{\text{red}}$ .

The *augmented base locus* of an  $\mathbf{R}$ -divisor  $D$  is defined to be

$$\mathbf{B}_+(D) := \bigcap_A \mathbf{B}(D - A),$$

where the intersection is over all ample divisors  $A$  such that  $D - A$  is a  $\mathbf{Q}$ -divisor. Equivalently, we have

$$\mathbf{B}_+(D) = \bigcap_{D=A+E} \text{Supp}(E),$$

where we take the intersection over all decompositions  $D = A + E$ , with  $A$  ample and  $E$  effective. It follows from definition that  $\mathbf{B}_+(D)$  is a closed subset of  $X$ , and  $\mathbf{B}_+(D) \neq X$  if and only if  $D$  is big. Moreover, there is  $\eta > 0$  such that for every ample divisor  $A$  with  $\|A\| < \eta$  and such that  $D - A$  is a  $\mathbf{Q}$ -divisor, we have  $\mathbf{B}_+(D) = \mathbf{B}(D - A)$ . It is easy to see that if  $D \equiv E$  then  $\mathbf{B}_+(D) = \mathbf{B}_+(E)$ . For a detailed study of augmented base loci, see [ELMNP2] §1. In addition, we will need the following property.

**Proposition 1.1.** *If  $D$  is a  $\mathbf{Q}$ -divisor, then  $\mathbf{B}(D)$  has no isolated points. In particular, for every  $\mathbf{R}$ -divisor  $D$ , the augmented base locus  $\mathbf{B}_+(D)$  has no isolated points.*

*Proof.* Suppose that  $x$  is an isolated point in  $\mathbf{B}(D)$ , and let  $m$  be large and divisible enough such that  $mD$  is integral and  $\mathbf{B}(D) = \text{Bs}(|mD|)_{\text{red}}$ . Let  $\mathfrak{a} \subseteq \mathcal{O}_X$  be the ideal defining the scheme  $\text{Bs}(|mD|) \setminus \{x\}$  and let  $f: X' \rightarrow X$  be the normalized blow-up along  $\mathfrak{a}$ . We can write  $f^*(mD) = M + F$ , where  $f^{-1}(\mathfrak{a}) = \mathcal{O}(-F)$  and the base locus of  $|M|$  is concentrated at the point  $f^{-1}(x)$ . A result of Zariski (see [Zar], and also [Ein]) implies that there is  $p$  such that  $|pM|$  is base-point free. Therefore  $x$  is not in the base locus of  $|pmD|$ , a contradiction.

If  $D$  is an  $\mathbf{R}$ -divisor, then  $\mathbf{B}_+(D) = \mathbf{B}(D - A)$  for some ample divisor  $A$  such that  $D - A$  is a  $\mathbf{Q}$ -divisor, so the last assertion follows.  $\square$

**Remark 1.2.** The assertion on  $\mathbf{B}_+(D)$  in the above proposition can also be proved using the elementary theory of multiplier ideals, avoiding the appeal to Zariski's theorem.

## 2. RESTRICTED VOLUMES AND ASYMPTOTIC INTERSECTION NUMBERS

**Restricted volumes.** Recall that  $X$  is a smooth projective variety. For any line bundle  $L$  on  $X$  and any subvariety  $V \subseteq X$ , we set

$$H^0(X|V, L) := \operatorname{Im}\left(H^0(X, L) \longrightarrow H^0(V, L|_V)\right),$$

while  $h^0(X|V, L)$  is the dimension of  $H^0(X|V, L)$ . Of course we use the analogous notation for divisors.

**Definition 2.1** (Restricted volume). If  $L$  is a line bundle on  $X$ , and if  $V \subseteq X$  is a subvariety of dimension  $d \geq 1$ , then the *restricted volume* of  $L$  along  $V$  is

$$(2) \quad \operatorname{vol}_{X|V}(L) = \limsup_{m \rightarrow \infty} \frac{h^0(X|V, mL)}{m^d/d!}.$$

Again, the same definition applies to divisors.

Note that if  $V = X$ , then the restricted volume of  $L$  along  $V$  is the usual volume of  $L$ , denoted by  $\operatorname{vol}_X(L)$ , or simply by  $\operatorname{vol}(L)$ . We refer to [Laz] §2.2.C for a study of the volume function. Our main goal in this section is to extend these results to the case of an arbitrary subvariety  $V \subseteq X$ . As we will see, everything goes over provided we assume that  $V \not\subseteq \mathbf{B}_+(L)$ . To begin with, the following lemma implies immediately that

$$\operatorname{vol}_{X|V}(qL) = q^d \operatorname{vol}_{X|V}(L),$$

so we can also define in the obvious way  $\operatorname{vol}_{X|V}(D)$  when  $D$  is a  $\mathbf{Q}$ -divisor.

**Lemma 2.2.** *Let  $D$  be any divisor on  $X$  and  $q \in \mathbf{N}$  a fixed positive integer. Then*

$$\limsup_{m \rightarrow \infty} \frac{h^0(X|V, mD)}{m^d/d!} = \limsup_{m \rightarrow \infty} \frac{h^0(X|V, qmD)}{(qm)^d/d!}.$$

*Proof.* The proof is identical to that of the corresponding statement for the usual volume function of a line bundle given in [Laz] Lemma 2.2.38.  $\square$

**Example 2.3** (Ample and nef divisors). If  $D$  is an ample  $\mathbf{Q}$ -divisor, then Serre vanishing implies  $\operatorname{vol}_{X|V}(D) = \operatorname{vol}_V(D|_V) = (D^d \cdot V)$ . We will see later that the same thing is true if  $D$  is only nef under the hypothesis  $V \not\subseteq \mathbf{B}_+(D)$  (cf. Corollary 2.17 and Example 5.5).

**Lemma 2.4.** *Let  $f : X' \longrightarrow X$  be a proper, birational morphism of smooth varieties and let  $D$  be a  $\mathbf{Q}$ -divisor on  $X$ . If  $V' \subseteq X'$  and  $V = f(V')$  have the same dimension, then*

$$\operatorname{vol}_{X'|V'}(f^*(D)) = \operatorname{vol}_{X|V}(D).$$

*Proof.* It is enough to note that for every  $m$  such that  $mD$  is an integral divisor, we have the commutative diagram

$$\begin{array}{ccc} H^0(X, mD) & \xrightarrow{u} & H^0(X', mf^*(D)) \\ \downarrow & & \downarrow \\ H^0(V, mD|_V) & \xrightarrow{v} & H^0(V', mf^*(D)|_{V'}), \end{array}$$

where  $u$  is an isomorphism and  $v$  is a monomorphism.  $\square$

**Remark 2.5.** Note by contrast that

$$\mathrm{vol}_{V'}(f^*(D)|_{V'}) = \deg(V' \longrightarrow V) \cdot \mathrm{vol}_V(D|_V).$$

**Asymptotic intersection numbers.** Let  $D$  be a  $\mathbf{Q}$ -divisor. Note that if  $V \subseteq \mathbf{B}(D)$ , then clearly  $\mathrm{vol}_{X|V}(D) = 0$ . Assume now that  $V \not\subseteq \mathbf{B}(D)$ . We can then define another invariant, an asymptotic intersection number of  $D$  and  $V$ , in the following way. Fix a natural number  $m > 0$  which is sufficiently divisible so that  $\mathbf{B}(D) = \mathrm{Bs}(|mD|)_{\mathrm{red}}$ , and let

$$\pi_m : X_m \rightarrow X$$

be a resolution of the base ideal  $\mathfrak{b}_m = \mathfrak{b}(|mD|)$ . Thus we have a decomposition

$$\pi_m^*(|mD|) = |M_m| + E_m,$$

where  $M_m$  (the *moving part* of  $|mD|$ ) is free, and  $E_m$  is the *fixed part*. We can — and without further mention, always will — choose all such resolutions with the property that they are isomorphisms over the generic point of  $V$ . We then denote by  $\tilde{V}_m$  the proper transform of  $V$ , which by hypothesis is not contained in  $\mathrm{Supp}(E_m)$ .

**Definition 2.6** (Asymptotic intersection number). With the notation just introduced, the *asymptotic intersection number* of  $D$  and  $V$  is defined to be

$$\|D^d \cdot V\| := \limsup_{m \rightarrow \infty} \frac{(M_m^d \cdot \tilde{V}_m)}{m^d}.$$

Naturally enough, we make the analogous definition for line bundles.

**Remark 2.7.** The intersection numbers with  $M_m$  have the following interpretation: if  $D_1, \dots, D_d$  are general divisors in  $|mD|$ , then  $(M_m^d \cdot \tilde{V}_m)$  is equal to the number of points in  $D_1 \cap \dots \cap D_d \cap V$  that do not lie in  $\mathrm{Bs}(|mD|)$ . In particular, this number does not depend on the resolution we are choosing.

**Remark 2.8.** Another sort of asymptotic intersection number, involving the moving intersection points of  $n$  different big line bundles on  $X$ , is introduced and studied in [BDPP]. Under suitable conditions on the position of  $V$  with respect to the relevant base-loci, one could combine the two lines of thought to define an asymptotic intersection number of the type

$$\|L_1 \cdot \dots \cdot L_d \cdot V\|.$$

However we do not pursue this here.

**Remark 2.9.** We note that  $\|D^d \cdot V\|$  computes in fact the *limit*  $\lim_{m \rightarrow \infty} \frac{(M_m^d \cdot \tilde{V}_m)}{m^d} = \sup_m \frac{(M_m^d \cdot \tilde{V}_m)}{m^d}$ , where the limit and the supremum are over all  $m$  such that  $\mathbf{B}(D) = \mathrm{Bs}(|mD|)_{\mathrm{red}}$ . Indeed, given such  $p$  and  $q$ , we may take  $\pi : X' \rightarrow X$  that satisfies our requirements for  $|pD|$ ,  $|qD|$  and  $|(p+q)D|$ . If  $\tilde{V}$  is the proper transform of  $V$  on  $X'$ , we deduce that  $M_{p+q} - (M_p + M_q)$  is effective and does not contain  $\tilde{V}$  in its support, so

$$(M_{p+q}^d \cdot \tilde{V})^{1/d} \geq ((M_p + M_q)^d \cdot \tilde{V})^{1/d} \geq (M_p^d \cdot \tilde{V})^{1/d} + (M_q^d \cdot \tilde{V})^{1/d},$$

where for the last inequality we refer to [Laz], Corollary 1.6.3. It is standard to deduce our claim from this inequality.

**Remark 2.10.** It follows from the previous remark that  $\|(mD)^d \cdot V\| = m^d \|D^d \cdot V\|$  for every  $m$ .

The next result gives another interpretation of these intersection numbers.

**Proposition 2.11.** *If  $D$  is a  $\mathbf{Q}$ -divisor, and if  $V$  is not contained in  $\mathbf{B}_+(D)$ , then*

$$(3) \quad \|D^d \cdot V\| = \sup_{\pi^*D=A+E} (A^d \cdot \tilde{V}),$$

where the supremum is taken over all projective birational morphisms  $\pi : X' \rightarrow X$  with  $X'$  smooth, that give an isomorphism at the general point of  $V$ , and over all expressions  $\pi^*D = A+E$ , where  $A$  and  $E$  are  $\mathbf{Q}$ -divisors, with  $A$  ample,  $E$  effective and  $\tilde{V} \not\subseteq \text{Supp}(E)$ . (Here  $\tilde{V}$  denotes the proper transform of  $V$ .)

*Proof.* Consider first any morphism  $\pi$  as in the statement of the proposition and let  $m$  be divisible enough. The number of points outside  $\text{Bs}(|mD|)$  that lie on the intersection of  $d$  general members of  $|mD|$  with  $V$  is the same as the number of points outside  $\text{Bs}(|\pi^*(mD)|) = \pi^{-1}(\text{Bs}(|mD|))$  that lie on the intersection of  $d$  general members of  $|\pi^*(D)|$  with  $\tilde{V}$ . Moreover, since  $\tilde{V}$  is not contained in  $\text{Supp}(E)$ , this number is at least the number of intersection points of  $\tilde{V}$  with  $d$  general members in  $|mA|$ , which is  $m^d(A^d \cdot \tilde{V})$ . Dividing by  $m^d$  and letting  $m$  go to infinity gives the inequality “ $\geq$ ” in the statement. On the other hand, by definition we have

$$\|D^d \cdot V\| := \limsup_{m \rightarrow \infty} \frac{(M_m^d \cdot \tilde{V}_m)}{m^d}.$$

It is easy to see that since  $V \not\subseteq \mathbf{B}_+(D)$ , we have  $\tilde{V}_m \not\subseteq \mathbf{B}_+(M_m)$ . Therefore we can write  $M_m = A + E$ , with  $A$  ample,  $E$  effective, and  $\tilde{V}_m \not\subseteq \text{Supp}(E)$ . If  $p \in \mathbf{N}^*$ , then we have  $M_m = (1/p)E + A_p$ , where  $A_p = \frac{1}{p}A + \frac{p-1}{p}M_m$  is ample since  $M_m$  is nef. The opposite inequality in the statement follows from  $\lim_{p \rightarrow \infty} (A_p^d \cdot \tilde{V}_m) = (M_m^d \cdot \tilde{V}_m)$ .  $\square$

As the right hand side of (3) depends only on the numerical class of  $D$ , we deduce the following:

**Corollary 2.12.** *If  $D_1 \equiv D_2$  are  $\mathbf{Q}$ -divisors, and if  $V \not\subseteq \mathbf{B}_+(D_1)$ , then*

$$\|D_1^d \cdot V\| = \|D_2^d \cdot V\|.$$

**A generalized Fujita Approximation Theorem.** The next result shows that if  $V$  is not contained in  $\mathbf{B}_+(D)$ , then the two invariants we have defined for  $D$  along  $V$  are the same. In the case  $V = X$ , this is Fujita’s Approximation Theorem (see [DEL]). In addition, we give a formula for the restricted volume in terms of asymptotic multiplier ideals, connecting our approach to ideas for defining invariants due to Tsuji [Ts2]. We mention that the relationship between asymptotic intersection numbers and the growth of sections vanishing along restricted multiplier ideals appears also in the recent work of Takayama [Ta2]. Note that these statements are interesting only for big divisors  $D$ , since otherwise  $\mathbf{B}_+(D) = X$ . We will assume familiarity with the basic theory of multiplier ideals developed in Part III of [Laz].



If  $D$  is an integral divisor, we denote by  $\mathcal{J}(\|mD\|)$  the asymptotic multiplier ideal of  $mD$ . For simplicity, we use the following notation: if  $\mathcal{I}$  is an ideal sheaf on a variety  $X$ , and if  $V \subseteq X$  is a subvariety, then  $\mathcal{I}|_V$  denotes the ideal  $\mathcal{I} \cdot \mathcal{O}_V$ .

**Theorem 2.13.** *Let  $D$  be a  $\mathbf{Q}$ -divisor on  $X$ , and let  $V$  be a  $d$ -dimensional subvariety of  $X$  ( $d \geq 1$ ) such that  $V \not\subseteq \mathbf{B}_+(D)$ . Then*

$$\mathrm{vol}_{X|V}(D) = \|D^d \cdot V\| = \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD) \otimes \mathcal{J}(\|mD\|)|_V)}{m^d/d!},$$

where in the last term we take the limit over  $m$  sufficiently divisible so that  $mD$  is integral.

The proof of the above theorem will be given in the next section. We record now several consequences and examples. Theorem 2.13 together with Corollary 2.12 imply the following:

**Corollary 2.14.** *If  $D$  is a  $\mathbf{Q}$ -divisor and if  $V$  is a subvariety such that  $V \not\subseteq \mathbf{B}_+(D)$ , then the restricted volume  $\mathrm{vol}_{X|V}(D)$  depends only on the numerical class of  $D$ .*

**Corollary 2.15.** *If  $D$  is a  $\mathbf{Q}$ -divisor on  $X$  and  $V \subseteq X$  is a  $d$ -dimensional subvariety such that  $V \not\subseteq \mathbf{B}_+(D)$ , then*

$$\limsup_{m \rightarrow \infty} \frac{h^0(X|V, mD)}{m^d/d!} = \lim_{m \rightarrow \infty} \frac{h^0(X|V, mD)}{m^d/d!}.$$

In other words, the restricted volume is actually the limit

$$\mathrm{vol}_{X|V}(D) = \lim_{m \rightarrow \infty} \frac{h^0(X|V, mD)}{m^d/d!}.$$

*Proof.* This fact is an immediate consequence of Theorem 2.13 together with Proposition 2.11, the proof being identical to that of the corresponding statement for the usual volume ([Laz], Example 11.4.7).  $\square$

**Corollary 2.16.** *Let  $D_1$  and  $D_2$  be two  $\mathbf{Q}$ -divisors and  $V \subseteq X$  a subvariety of dimension  $d \geq 1$  such that  $V \not\subseteq \mathbf{B}_+(D_1) \cup \mathbf{B}_+(D_2)$ . Then*

$$\mathrm{vol}_{X|V}(D_1 + D_2)^{1/d} \geq \mathrm{vol}_{X|V}(D_1)^{1/d} + \mathrm{vol}_{X|V}(D_2)^{1/d}.$$

*Proof.* The assertion follows from Theorem 2.13 and Proposition 2.11, using the corresponding concavity property for the volumes of ample line bundles (see [Laz], Corollary 1.6.3).  $\square$

**Corollary 2.17.** *Suppose that  $D$  is a nef  $\mathbf{Q}$ -divisor and that  $V \subseteq X$  is a subvariety of dimension  $d \geq 1$ . If  $V$  is not contained in  $\mathbf{B}_+(D)$ , then  $\mathrm{vol}_{X|V}(D) = (D^d \cdot V)$ .*

*Proof.* We may assume that  $D$  is an integral divisor. Since  $D$  is nef and big, we have

$$\mathcal{J}(X, \|mD\|) = \mathcal{O}_X \text{ for all } m$$

([Laz], Proposition 11.2.18). It follows from Theorem 2.13 that

$$\mathrm{vol}_{X|V}(D) = \mathrm{vol}_V(D|_V) = (D^d \cdot V),$$

where we use the corresponding result for the usual volume function ([Laz], Corollary 1.4.41).  $\square$

**Remark 2.18.** If  $D$  is a nef divisor, then  $\mathbf{B}_+(D) = \text{Null}(D)$ , where  $\text{Null}(D)$  is the union of the subvarieties  $V$  of  $X$  such that  $D|_V$  is not big. This is the main result of [Na1], which we will reprove in Corollary 5.6 below, allowing also  $\mathbf{R}$ -coefficients.

**Example 2.19.** Suppose now that  $D$  is a pseudo-effective  $\mathbf{Q}$ -divisor on a surface  $X$ . Recall that  $D$  has a Zariski decomposition  $D = P + N$ , where  $P$  and  $N$  are  $\mathbf{Q}$ -divisors, with  $P$  nef and  $N$  effective, inducing for all divisible enough  $m$  isomorphisms:

$$(4) \quad H^0(X, mP) \simeq H^0(X, mD)$$

(see [Bă] for details). It is shown in [ELMNP2], Example 1.11, that we have  $\text{Supp}(N) \subseteq \mathbf{B}_+(D) = \mathbf{B}_+(P) = \text{Null}(P)$ .

If  $C$  is an irreducible curve on  $X$ , and if  $C \not\subseteq \text{Supp}(N)$ , then (4) induces an equality  $\text{vol}_{X|C}(D) = \text{vol}_{X|C}(P)$ . If, moreover,  $C$  is not contained in  $\text{Null}(P)$ , then Corollary 2.17 gives  $\text{vol}_{X|C}(D) = (P \cdot C)$ .

A theorem of Angehrn and Siu [AS] on effective base-point freeness for line bundles of the form  $K_X + L$ , with  $L$  ample, can be extended to the case of arbitrary big divisors, as follows.

**Theorem 2.20.** *Let  $L$  be a line bundle on a smooth, projective  $n$ -dimensional variety  $X$ . If  $x \notin \mathbf{B}_+(L)$  is such that for every positive-dimensional subvariety  $V$  through  $x$  we have*

$$\text{vol}_{X|V}(L) > M^{\dim(V)},$$

*where  $M = \binom{n+1}{2}$ , then  $x$  is not in the base locus of  $K_X + L$ .*

We do not give the proof of this statement, as it is rather straightforward, combining the method of Angehrn and Siu (see also Theorem 10.4.2 in [Laz]) with our generalized Fujita Approximation. We mention also that one can give similar uniform bounds that imply that  $K_X + L$  separates two points.

**Remark 2.21.** If  $V$  is a  $d$ -dimensional subvariety of  $X$  that is contained in  $\mathbf{B}_+(D)$  but not in  $\mathbf{B}(D)$ , then both  $\text{vol}_{X|V}(D)$  and  $\|D^d \cdot V\|$  are defined, but they are not equal in general. Example 5.10 below gives a big globally generated line bundle for which  $\text{vol}_{X|C}(L) < \|L \cdot C\|$  for some curve  $C$  contained in  $\mathbf{B}_+(L)$ .

However, if  $C$  is a curve not contained in  $\mathbf{B}(D)$  and  $\text{vol}_{X|C}(D) = 0$ , then we have  $\|D \cdot C\| = 0$ . Indeed, if  $\pi_m: X_m \rightarrow X$  and  $\pi_m^*(mD) = M_m + E_m$  are as in the definition of asymptotic intersection numbers, then  $\tilde{C}_m$  is not contained in the support of  $E_m$ , so

$$\text{vol}_{X|C}(D) = \text{vol}_{X_m|\tilde{C}_m}(\pi_m^*(D)) \geq \frac{\text{vol}_{X_m|\tilde{C}_m}(M_m)}{m},$$

so the right hand side is zero. Using the diagram in the proof of Lemma 2.4, we deduce that  $\tilde{C}_m$  is contracted by the morphism defined by  $|M_m|$ , so  $(M_m \cdot \tilde{C}_m) = 0$  for every  $m$ .

**Example 2.22.** Inspired by ideas of Tsuji [Ts2], Takayama introduced in [Ta1] the following asymptotic intersection number of a  $\mathbf{Q}$ -divisor  $D$  with an arbitrary curve  $C \subset X$ :

$$\|D \cdot C\|' := \limsup_{m \rightarrow \infty} \frac{\deg(\mathcal{O}_C(mD) \otimes \mathcal{J}(\|mD\|)|_C)}{m},$$

where the degree of  $\mathcal{J}(\|mD\|)|_C$  is defined as the degree of the invertible sheaf  $\mathcal{J}(\|mD\|) \cdot \mathcal{O}_{C'}$  on the normalization  $\nu : C' \rightarrow C$ . One can show, using the multiplier ideal interpretation in the statement of Theorem 2.13, that if  $C \not\subseteq \mathbf{B}_+(D)$ , then  $\|D \cdot C\|' = \text{vol}_{X|C}(D)$ . If  $C \subseteq \mathbf{B}_+(D)$ , then we have only one inequality, namely  $\|D \cdot C\|' \geq \text{vol}_{X|C}(D)$ .

**Example 2.23** (Interpretation of Kodaira-Iitaka dimension). Let  $X$  be a smooth projective variety of dimension  $n$  and  $D$  a  $\mathbf{Q}$ -divisor on  $X$  such that its Iitaka dimension  $\kappa(D)$  is nonnegative. Define  $r(D)$  to be the largest nonnegative integer  $d$  such that through a very general point on  $X$  there is a  $d$ -dimensional irreducible subvariety  $V$  with the following property: for every curve  $C \subseteq V$  that is not contained in  $\mathbf{B}(D)$  we have  $\|D \cdot C\| = 0$ . Then we have  $r(D) = n - \kappa(D)$ . In order to show this, consider  $m$  divisible enough such that the rational map  $\phi_m$  to  $\mathbf{P}^N$  defined by the complete linear series  $|mD|$  satisfies  $\dim(\phi_m(X)) = \kappa(D)$ . With the usual notations, we also have a morphism  $\psi_m : X_m \rightarrow \mathbf{P}^N$  defined by the basepoint-free linear series  $|M_m|$ . Let  $C$  be a curve in  $X$  not contained in  $\mathbf{B}(D)$ . If  $\phi_m$  is defined at the generic point of  $C$ , then  $C$  is contracted by  $\phi_m$  if and only if its proper transform  $\tilde{C}_m$  is contracted by  $\psi_m$ . But this last statement is equivalent to  $(M_m \cdot \tilde{C}_m) = 0$ . In particular, this shows that if  $V$  is as in the definition of  $r(D)$ , then  $V$  is contracted by every  $\phi_m$ .

Consider now the Iitaka fibration corresponding to  $D$  (see [Laz], §2.1.C). This is a morphism of normal varieties (defined up to birational equivalence)  $\phi_\infty : X_\infty \rightarrow Y_\infty$  having connected fibers and such that for all  $m$  divisible enough we have a commutative diagram

$$\begin{array}{ccc} X_\infty & \xrightarrow{u_\infty} & X \\ \phi_\infty \downarrow & & \downarrow \phi_m \\ Y_\infty & \xrightarrow{u_m} & Y_m \end{array}$$

with  $u_\infty$  a birational morphism and  $u_m$  a birational map. Let  $U$  be the set of points  $x$  in  $X \setminus \mathbf{B}(D)$  such that  $u_\infty$  is an isomorphism over a neighborhood of  $x$  and  $\phi_\infty(u_\infty^{-1}(x))$  lies in an open subset on which  $u_m$  is an isomorphism. We see that if  $C$  is a curve in  $X$  that intersects  $U$  and  $\tilde{C} \subseteq X_\infty$  is its proper transform, then  $C$  is contracted by some  $\phi_m$  (with  $m$  divisible enough) if and only if  $\tilde{C}$  is contracted by  $\phi_\infty$ . Since this condition is independent on  $m$ , we see that it is satisfied if and only if  $\|D \cdot C\| = 0$ . Our formula for  $r(D)$  now follows easily.

We restate the conclusion of the above discussion as follows. Compare this with Theorem 1.3 in [Ta1], where a similar result is proved with  $\|D \cdot C\|$  replaced by  $\|D \cdot C\|'$ .

**Corollary 2.24.** *If  $\phi_\infty : X_\infty \rightarrow Y_\infty$  is the Iitaka fibration corresponding to  $D$  and if  $C$  is a curve through a very general point on  $X$ , then its proper transform  $\tilde{C} \subseteq X_\infty$  is*

contracted by  $\phi_\infty$  if and only if  $\|D \cdot C\| = 0$ . The analogous assertion holds if we replace the morphism  $\phi_\infty$  by the rational map on  $X$  defined by  $|mD|$ , with  $m$  divisible enough.

### 3. THE PROOF OF GENERALIZED FUJITA APPROXIMATION

For the proof of Theorem 2.13, we will need a few lemmas.

**Lemma 3.1.** *For any big  $\mathbf{Q}$ -divisor  $D$ , if  $\mathfrak{b}_m$  denotes the ideal defining the base locus of  $|mD|$ , then there exists an effective integral divisor  $G$  on  $X$  – which one may take to be very ample – such that*

$$\mathfrak{b}_m(-G) \subseteq \mathcal{J}(\|mD\|)(-G) \subseteq \mathfrak{b}_m,$$

for all  $m$  sufficiently large and divisible. If  $V \not\subseteq \mathbf{B}_+(D)$ , then  $G$  can be chosen such that  $V \not\subseteq \text{Supp}(G)$ .

*Proof.* The first statement appears in [Laz] Theorem 11.2.21, but we recall the construction in order to emphasize the second point. Specifically, let  $H$  be a very ample line bundle on  $X$ , and consider the ample line bundle  $A := K_X + (n+1)H$ , where  $n$  is the dimension of  $X$ . For  $a \gg 0$  sufficiently divisible,  $\mathcal{O}_X(aD - A)$  is a big line bundle with sections, and for any  $G$  in  $|aD - A|$  the sequence of inclusions in the Lemma holds.

Now note that since  $a$  is large and divisible enough,  $\text{Bs}(|aD - A|)$  is contained in  $\mathbf{B}_+(D)$ . Indeed, if  $m$  is large enough then  $\mathbf{B}_+(D) = \mathbf{B}(D - \frac{1}{m}D)$ , and if  $a$  is divisible enough, then

$$\mathbf{B}(D - \frac{1}{m}A) = \text{Bs}(|aD - \frac{a}{m}A|)_{\text{red}} \supseteq \text{Bs}(|aD - A|)_{\text{red}}.$$

Thus if  $V \not\subseteq \mathbf{B}_+(D)$ , then there exists  $G \in |aD - A|$  such that  $V \not\subseteq \text{Supp}(G)$ .  $\square$

**Lemma 3.2.** *Let  $L$  be a big semiample line bundle on  $X$  and suppose that  $V$  is a  $d$ -dimensional subvariety of  $X$  such that  $V \not\subseteq \mathbf{B}_+(L)$ . Then  $\text{vol}_{X|V}(L) = (L^d \cdot V)$ .*

*Proof.* By choosing  $k \gg 0$ , we may assume that the morphism  $f : X \rightarrow \mathbf{P}^N$  given by the linear series  $|kL|$  is birational onto its image, with trivial Stein factorization. Since  $V \not\subseteq \mathbf{B}_+(L)$ , we may also assume that the restriction of  $f$  to  $V$  is birational onto its image. The argument in the proof of Lemma 2.4 reduces us to the case of a very ample line bundle, when the equality is clear.  $\square$

**Lemma 3.3.** *If  $V$  is a subvariety of  $X$  of dimension  $d \geq 1$ , and if  $D$  is an integral divisor on  $X$ , then for every  $k$  we have*

$$(5) \quad \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD) \otimes \mathcal{J}(\|mD\|)|_V)}{m^d} = \limsup_{p \rightarrow \infty} \frac{h^0(V, \mathcal{O}(pkD) \otimes \mathcal{J}(\|pkD\|)|_V)}{k^d p^d}.$$

*Proof.* Let's denote the left hand side in (5) by  $L_1$  and the right hand side by  $L_2$ . We obviously have  $L_1 \geq L_2$  and we need to prove the reverse inequality. To this end let  $A$  be a very ample line bundle on  $V$ , and fix a very general divisor  $H \in |A|$ . Assuming as

we may that  $H$  doesn't contain any of the subvarieties defined by the associated primes of the ideal sheaves  $\mathcal{J}(\|mD\|)|_V$ , we have for every  $m \geq 0$  an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_V(mD) \otimes \mathcal{J}(\|mD\|)|_V &\xrightarrow{\cdot H} \mathcal{O}_V(mD + A) \otimes \mathcal{J}(\|mD\|)|_V \\ &\longrightarrow \mathcal{O}_H(mD + A) \otimes \mathcal{J}(\|mD\|)|_H \longrightarrow 0. \end{aligned}$$

Since in any event

$$h^0(H, \mathcal{O}_H(mD + A) \otimes \mathcal{J}(\|mD\|)|_H) \leq h^0(H, \mathcal{O}_H(mD + A)) = O(m^{d-1}),$$

we see in the first place that

$$(*) \quad L_1 = \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD + A) \otimes \mathcal{J}(\|mD\|)|_V)}{m^d}.$$

Now given  $k \geq 1$  and  $m \gg 0$ , write  $m = pk + \ell$  with  $0 \leq \ell \leq k - 1$ . Choose a very ample line bundle  $A$  which is sufficiently positive so that  $A + \ell D$  is very ample for each  $0 \leq \ell \leq k - 1$ , and fix a very general divisor  $H_\ell \in |\ell D + A|$ . As above we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_V(pkD) \otimes \mathcal{J}(\|pkD\|)|_V &\xrightarrow{\cdot H_\ell} \mathcal{O}_V(mD + A) \otimes \mathcal{J}(\|pkD\|)|_V \\ &\longrightarrow \mathcal{O}_{H_\ell}(mD + A) \otimes \mathcal{J}(\|pkD\|)|_{H_\ell} \longrightarrow 0. \end{aligned}$$

Noting that  $\mathcal{J}(\|mD\|) \subseteq \mathcal{J}(\|pkD\|)$ , we find as before that

$$\begin{aligned} L_2 &= \limsup_{p \rightarrow \infty} \frac{h^0(V, \mathcal{O}(pkD) \otimes \mathcal{J}(\|pkD\|)|_V)}{p^d k^d} \\ &= \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD + A) \otimes \mathcal{J}(\|(m - \ell)D\|)|_V)}{m^d} \\ &\geq \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD + A) \otimes \mathcal{J}(\|mD\|)|_V)}{m^d}. \end{aligned}$$

Together with (\*), this gives the required inequality  $L_2 \geq L_1$ .  $\square$

*Proof of Theorem 2.13.* Consider a common log resolution  $\pi_m : X_m \rightarrow X$  for the ideal  $\mathfrak{b}_m$  defining the base locus of  $|mD|$  and for  $\mathcal{I}(\|mD\|)$ . We denote  $\mathfrak{b}_m \cdot \mathcal{O}_{X_m} = \mathcal{O}_{X_m}(-E_m)$  and  $\mathcal{I}(\|mD\|) \cdot \mathcal{O}_{X_m} = \mathcal{O}_{X_m}(-F_m)$ , and also  $M_m := \pi_m^*(mD) - E_m$  and  $N_m := \pi_m^*(mD) - F_m$ . Since  $\mathfrak{b}_m \subseteq \mathcal{I}(\|mD\|)$ , we have  $E_m \geq F_m$ . Note also that  $|M_m|$  is the moving part of the linear series  $|\pi_m^*(mD)|$ , so it is basepoint-free. Moreover, since  $V \not\subseteq \mathbf{B}_+(D)$ , we have  $\tilde{V}_m \not\subseteq \mathbf{B}_+(M_m)$ .

We have, to begin with

$$\text{vol}_{X|V}(D) \geq \frac{\text{vol}_{X_m|\tilde{V}_m}(M_m)}{m^d} = \frac{\text{vol}(M_m|_{\tilde{V}_m})}{m^d} = \frac{M_m^d \cdot \tilde{V}_m}{m^d},$$

where the first inequality follows easily from the definition and Lemma 2.4, while the second and third equalities follow from Lemma 3.2. This implies that

$$\text{vol}_{X|V}(D) \geq \|D^d \cdot V\|.$$

On the other hand, since for any (divisible enough)  $m$  we have

$$H^0(X, \mathcal{O}(mD)) = H^0(X, \mathcal{O}(mD) \otimes \mathcal{I}(\|mD\|)),$$

we see immediately that

$$\limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD) \otimes \mathcal{I}(\|mD\|)|_V)}{m^d/d!} \geq \text{vol}_{X|V}(D).$$

To finish, it suffices then to prove

$$\|D^d \cdot V\| \geq \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD) \otimes \mathcal{I}(\|mD\|)|_V)}{m^d/d!}.$$

To this end, we first apply the inclusions

$$\mathfrak{b}_m(-G) \subseteq \mathcal{I}(\|mD\|)(-G) \subseteq \mathfrak{b}_m$$

given by Lemma 3.1. On  $X_m$ , this immediately implies:

$$(6) \quad \text{vol}((M_m + \pi_m^* G)|_{\tilde{V}_m}) \geq \text{vol}(N_m|_{\tilde{V}_m}) \geq \text{vol}(M_m|_{\tilde{V}_m}).$$

On the other hand, by pulling back to  $\tilde{V}_m$  in two different ways, we have:

$$h^0(\tilde{V}_m, kN_m|_{\tilde{V}_m}) \geq h^0(V, \mathcal{O}(kmD) \otimes \mathcal{I}(\|mD\|)^k|_V) \geq h^0(V, \mathcal{O}(kmD) \otimes \mathcal{I}(\|kmD\|)|_V),$$

where for the last inequality we use the Subadditivity Theorem (see [DEL] or [Laz], Theorem 11.2.3). Thus by multiplying the inequalities by  $d!/(km)^d$  and letting first  $k$  and then  $m$  go to  $\infty$ , we obtain:

$$(7) \quad \limsup_{m \rightarrow \infty} \frac{\text{vol}(N_m|_{\tilde{V}_m})}{m^d} \geq \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}(mD) \otimes \mathcal{I}(\|mD\|)|_V)}{m^d/d!},$$

thanks to Lemma 3.3.

Combining (6) and (7), we are then done if we show

$$(8) \quad \limsup_{m \rightarrow \infty} \frac{\text{vol}((M_m + \pi_m^* G)|_{\tilde{V}_m})}{m^d} = \limsup_{m \rightarrow \infty} \frac{\text{vol}(M_m|_{\tilde{V}_m})}{m^d}.$$

But since the bundles in question are nef, the volumes appearing here are computed as intersection numbers. Expanding out the one on the left, we find that (8) will follow if we show that

$$(9) \quad \limsup_{m \rightarrow \infty} \frac{(M_m^{d-i} \cdot (\pi_m^* G)^i \cdot \tilde{V}_m)}{m^d} = 0$$

when  $i > 0$ . But we can find a fixed ample line bundle  $A$  on  $X$  such that  $\pi^*(mA) - M_m$  is effective (simply take  $A$  such that  $A - L$  is effective). Then

$$(M_m^{d-i} \cdot (\pi_m^* G)^i \cdot \tilde{V}_m) \leq ((\pi_m^*(mA))^{d-i} \cdot (\pi_m^* G)^i \cdot \tilde{V}_m),$$

and (9) follows.  $\square$

## 4. APPROXIMATING RESTRICTED VOLUMES VIA JET SEPARATION

We start with a simple lemma which shows that separating jets at a set of points on a subvariety gives a lower bound for the restricted volume. If  $x$  is a point on a subvariety  $W$  of  $X$ , we denote by  $\mathfrak{m}_{W,x}$  the ideal defining  $x$  in the local ring  $\mathcal{O}_{W,x}$  of  $W$  at  $x$ . If  $x_1, \dots, x_N$  are points on  $X$ , then we say that a line bundle  $L$  on  $X$  simultaneously separates  $p_i$ -jets at each  $x_i$  if the map

$$H^0(X, L) \longrightarrow \bigoplus_{i=1}^N H^0(X, L \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{X,x_i}^{p_i+1})$$

is surjective.

**Lemma 4.1.** *Let  $L$  be a line bundle on the smooth variety  $X$ , and let  $V \subseteq X$  be a subvariety of dimension  $d \geq 1$ . If  $x_1, \dots, x_N$  are distinct points on  $V$  such that  $L$  simultaneously separates  $p_i$ -jets on  $X$  at each of the points  $x_i$ , then*

$$\mathrm{vol}_{X|V}(L) \geq \sum_{i=1}^N \mathrm{mult}_{x_i} V \cdot p_i^d.$$

*Proof.* If  $L$  separates  $p_i$ -jets at each  $x_i$ , then by taking polynomials in sections one sees that if  $m \geq 1$  then  $mL$  separates  $mp_i$ -jets on  $X$  at  $x_i$  for all  $i$ . Moreover, for every such  $m$  we have a commutative diagram

$$\begin{array}{ccc} H^0(X, mL) & \longrightarrow & \bigoplus_i H^0(X, mL \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{X,x_i}^{mp_i+1}) \\ \downarrow & & \downarrow \\ H^0(V, mL|_V) & \longrightarrow & \bigoplus_i H^0(V, mL \otimes \mathcal{O}_{V,x_i}/\mathfrak{m}_{V,x_i}^{mp_i+1}). \end{array}$$

As the right vertical map is evidently surjective, we deduce that

$$\dim_{\mathbf{C}} \mathrm{Im} H^0(X|V, mL) \geq \sum_{i=1}^N \dim_{\mathbf{C}} \mathcal{O}_{V,x_i}/\mathfrak{m}_{V,x_i}^{mp_i+1},$$

and the lemma follows.  $\square$

We now prove the converse to Lemma 4.1, showing that we can approximate restricted volumes by separation of jets at general points on the subvariety. It is convenient to make the following definition. Let  $D$  be a  $\mathbf{Q}$ -divisor on the smooth projective variety  $X$ , and let  $V$  be a subvariety of  $X$  of dimension  $d \geq 1$ . For every positive integer  $N$ , let  $\epsilon_V(D, N)$  be the supremum of the set of nonnegative rational numbers  $t$  with the property that for some  $m$  with  $mt \in \mathbf{Z}$  and  $mD$  an integral divisor,  $mD$  simultaneously separates  $mt$ -jets at every general set of points  $x_1, \dots, x_N \in V$  (if there is no such  $t$ , then we put  $\epsilon_V(D, N) = 0$ ). Note that with this notation, Lemma 4.1 implies  $\mathrm{vol}_{X|V}(D) \geq N \cdot \epsilon_V(D, N)^d$ .

**Theorem 4.2.** *If  $D$  is a  $\mathbf{Q}$ -divisor on the smooth projective variety  $X$ , and  $V \subseteq X$  is a subvariety of dimension  $d \geq 1$  such that  $V \not\subseteq \mathbf{B}_+(D)$ , then*

$$\sup_{N \geq 1} N \cdot \epsilon_V(D, N)^d = \limsup_{N \rightarrow \infty} N \cdot \epsilon_V(D, N)^d = \mathrm{vol}_{X|V}(D).$$

**Remark 4.3.** The above statement is inspired by a key step in the proof of Fujita's Approximation Theorem from [Na2]. In *loc. cit.* one proves a variant of this statement when  $V = X$ . However, we take the opposite approach, and we deduce Theorem 4.2 from our generalization of Fujita's Theorem to restricted volumes.

*Proof of Theorem 4.2.* We need to show that for every  $\delta > 0$  we can find arbitrarily large values of  $N$  such that for a suitable positive  $\epsilon \in \mathbf{Q}$  with  $N\epsilon^d > \text{vol}_{X|V}(D) - \delta$ , and for some positive integer  $m$  such that  $m\epsilon \in \mathbf{Z}$  and  $mD$  is an integral divisor,  $mD$  simultaneously separates  $m\epsilon$ -jets on  $X$  at any general points  $x_1, \dots, x_N \in V$ . Suppose first that we know this when  $D$  is ample. Since  $V \not\subseteq \mathbf{B}_+(D)$ , it follows from Theorem 2.13 and Proposition 2.11 that we can find a proper morphism  $\pi : X' \rightarrow X$  that is an isomorphism over the generic point of  $V$ , and a decomposition  $\pi^*D = A + E$ , with  $A$  ample,  $E$  effective and  $\tilde{V} \not\subseteq \text{Supp}(E)$  such that  $(A^d \cdot \tilde{V}) > \text{vol}_{X|V}(D) - \delta/2$  (we have denoted by  $\tilde{V}$  the proper transform of  $V$ ). We apply the ample case for  $A$ ,  $\tilde{V}$  and  $\delta/2$  to get  $\epsilon$ ,  $N$  and  $m$ . We may clearly assume also that  $mE$  is integral. If  $x_1, \dots, x_N \in \tilde{V}$  are general points (in particular they do not lie on the union of the support of  $E$  with the exceptional locus of  $\pi$ ), and if we identify the  $x_i$  with their projections to  $X$ , then we have a commutative diagram

$$\begin{array}{ccc} H^0(X', \mathcal{O}(mA)) & \xrightarrow[\psi_1]{} & \bigoplus_{i=1}^N H^0(X', \mathcal{O}(mA) \otimes \mathcal{O}_{X', x_i} / \mathfrak{m}_{X', x_i}^{m\epsilon+1}) \\ \downarrow \phi_1 & & \downarrow \phi_2 \\ H^0(X, \mathcal{O}(mD)) & \xrightarrow[\psi_2]{} & \bigoplus_{i=1}^N H^0(X, \mathcal{O}(mD) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{X, x_i}^{m\epsilon+1}) \end{array}$$

where  $\phi_1$  and  $\phi_2$  are induced by multiplication with the section defining  $mE$ . Hence  $\phi_2$  is an isomorphism, and since  $\psi_1$  is surjective,  $\psi_2$  is surjective, too. Therefore we get our statement for  $D$ ,  $V$  and  $\delta$ .

It follows that in order to prove the theorem we may assume that  $D$  is ample, in which case  $\text{vol}_{X|V}(D) = (D^d \cdot V)$ . Moreover, by replacing  $D$  with a suitable power, we may suppose that it is very ample.

We make a parenthesis to recall the following well-known fact. Suppose that  $L$  is an ample line bundle on a variety  $X$ , and suppose that  $\Gamma = \{x_1, \dots, x_M\}$  is a set of smooth points on  $X$ . Let  $f : X' \rightarrow X$  be the blowing-up along  $\Gamma$  with exceptional divisor  $F = \sum_{i=1}^M F_i$ . If  $\beta > 0$ , then  $f^*(L) - \beta F$  is nef if and only if for every positive rational number  $\epsilon < \beta$ , if  $k$  is divisible enough, then  $kL$  separates  $k\epsilon$ -jets at  $x_1, \dots, x_M$ . Moreover, this is the case if and only if for every irreducible curve  $C$  in  $X$  we have

$$(L \cdot C) \geq \beta \sum_{i=1}^M \text{mult}_{x_i}(C).$$

If this holds, and  $\beta' < \beta$ , then  $f^*(L) - \beta' F$  is ample on  $X'$ . Note also that if the condition is satisfied for  $\Gamma$ , then it is satisfied for any subset  $\Gamma' \subseteq \Gamma$  too. We take  $p$  such that  $V$  is cut out by equations in  $|pD|$ . Let  $H_{d+1}, \dots, H_n \in |pD|$  be general elements vanishing on  $V$ , so  $V$  is an irreducible component of  $W := \bigcap_{i=d+1}^n H_i$ . Moreover,  $V = W$  scheme-theoretically



at the generic point of  $V$ , and  $W \setminus V$  is smooth of dimension  $d$ . Let  $H_1, \dots, H_d$  be general elements in  $|pD|$ , so the following sets

$$\Gamma' := V \cap H_1 \cap \dots \cap H_d \subseteq \Gamma := \bigcap_{i=1}^n H_i$$

are smooth and zero-dimensional. Let  $x_1, \dots, x_M$  be the points in  $\Gamma$ , and suppose that they are numbered such that the first  $N$  are the points in  $\Gamma'$ , where  $N = p^d(D^d \cdot V)$ . Note that if  $C \subseteq X$  is an irreducible curve, then there is  $i \leq n$  such that  $C \not\subseteq H_i$ . It follows from Bézout's theorem that

$$(H_i \cdot C) \geq \sum_{j=1}^M \text{mult}_{x_j}(C).$$

We deduce from the previous discussion that if  $\epsilon < 1/p$  and  $m$  is divisible enough, then  $mD$  separates  $m\epsilon$ -jets at  $x_1, \dots, x_M$ , hence at  $x_1, \dots, x_N$ . Given  $\delta$  and  $\eta > 0$ , we choose  $p \gg 0$  as above and such that  $1/p < \eta$ . If  $\epsilon$  is such that  $1 > (p\epsilon)^d > 1 - \frac{\delta}{(D^d \cdot V)}$ , we see that for  $m$  divisible enough the points  $x_1, \dots, x_N$  satisfy our requirement. It is now standard (using the behavior of ampleness in families) to deduce that the same property holds for any general set of points in  $V$ . We end by noting that since  $p$  can be taken arbitrarily large, the same is true for  $N$ .  $\square$

For future reference, we recall the following well-known facts.

**Remark 4.4.** Suppose that  $B$  is an ample  $\mathbf{Q}$ -divisor and  $x_1, \dots, x_N \in X$  are such that for  $m$  divisible enough,  $mB$  simultaneously separates  $m\epsilon'$ -jets at  $x_1, \dots, x_N$ . If  $\epsilon < \epsilon'$  and if  $m$  is divisible enough, then the linear system

$$\{P \in |mB| \mid \text{ord}_{x_i}(P) \geq m\epsilon + 1 \text{ for all } i\}$$

induces a base-point free linear system on  $X \setminus \{x_1, \dots, x_N\}$ .

Indeed, if  $\pi: X' \rightarrow X$  is the blowing-up at  $x_1, \dots, x_N$  with exceptional divisors  $E_1, \dots, E_N$ , our hypothesis implies that  $B' := \pi^*B - \epsilon(E_1 + \dots + E_N)$  is ample. Hence every  $m$  such that  $mB'$  is integral and globally generated satisfies our requirement.

**Remark 4.5.** Let  $B, x_1, \dots, x_N$  and  $\epsilon, \epsilon'$  be as in the previous remark. For every integral divisor  $H$  on  $X$ , if  $m$  is divisible enough, then  $(mB - H)$  separates  $m\epsilon$ -jets at  $x_1, \dots, x_N$ . Indeed, if  $\pi$  is as before, then  $\pi^*B - \epsilon(E_1 + \dots + E_N)$  is ample. It follows that if  $m$  is divisible enough, then  $mB - H$  is an integral divisor,  $m\epsilon \in \mathbf{Z}$  and  $\pi^*(mB - H) - m\epsilon(E_1 + \dots + E_N)$  is globally generated. This implies that  $(mB - H)$  simultaneously separates  $m\epsilon$ -jets at  $x_1, \dots, x_N$ .

**Remark 4.6.** Suppose now that  $D$  and  $V$  are as in Theorem 4.2, and that  $H$  is an integral effective divisor on  $X$ . If  $N \geq 1$  and  $\epsilon < \epsilon_V(D, N)$ , then for  $m$  divisible enough both  $mD$  and  $mD - H$  simultaneously separate  $m\epsilon$ -jets on  $X$  at every general set of points  $x_1, \dots, x_N$  in  $V$ . Indeed, we argue as in the proof of the theorem: we use Theorem 2.13 to reduce ourselves to the case of an ample line divisor  $A$  on some model  $X'$  over  $X$ . We apply for  $A$  the argument in the previous remark, and use the fact that if  $(mA - \pi^*(H))$  separates jets, then so does  $mD - H$ .

5. COMPONENTS OF  $\mathbf{B}_+$  AND THE RESTRICTED VOLUME FUNCTION

Given an  $\mathbf{R}$ -divisor  $D$ , we have  $\mathbf{B}_+(D') \subseteq \mathbf{B}_+(D)$  for every  $\mathbf{R}$ -divisor  $D'$  in a suitable open neighborhood of  $D$ . It follows that given a subvariety  $V$  of  $X$ , the set  $\{D \mid V \not\subseteq \mathbf{B}_+(D)\}$  is an open subset of the big cone.

**Definition 5.1.** Given a subvariety  $V \subseteq X$ , we denote by  $\text{Big}^V(X)_{\mathbf{R}}$  the set of big  $\mathbf{R}$ -divisor classes  $D$  such that  $V$  is not a proper subset of an irreducible component of  $\mathbf{B}_+(D)$ . It is clear that  $\text{Big}^V(X)_{\mathbf{R}}$  is an open convex subcone of the big cone.

The behavior of the restricted volumes as functions on subsets of the big cone can be summarized in the following theorem.

**Theorem 5.2.** (a) *If  $V$  is a fixed subvariety of a smooth projective variety  $X$ , with  $d = \dim(V) \geq 1$ , then the map*

$$\xi \longrightarrow \text{vol}_{X|V}(\xi)$$

*defined on the set of  $\mathbf{Q}$ -divisor classes  $\xi$  such that  $V \not\subseteq \mathbf{B}_+(\xi)$  is continuous and can be extended to a continuous function on the open set of all such  $\mathbf{R}$ -divisor classes. Moreover, it satisfies the concavity relation*

$$\text{vol}_{X|V}(D_1 + D_2)^{1/d} \geq \text{vol}_{X|V}(D_1)^{1/d} + \text{vol}_{X|V}(D_2)^{1/d}$$

*for every  $\mathbf{R}$ -classes as above.*

(b) *If  $D$  is a  $\mathbf{Q}$ -divisor such that  $V$  is an irreducible component of  $\mathbf{B}_+(D)$ , then  $\text{vol}_{X|V}(D) = 0$ . Moreover, if we put  $\text{vol}_{X|V}(\xi) = 0$  for every  $\xi \in \text{Big}^V(X)_{\mathbf{R}}$  such that  $V \subseteq \mathbf{B}_+(\xi)$ , then the function  $\xi \longrightarrow \text{vol}_{X|V}(\xi)$  is continuous over the entire cone  $\text{Big}^V(X)_{\mathbf{R}}$ .*

*Proof.* The proof of part (a) is quite standard, and we present it in what follows. Part (b) is the main technical result of the paper, and we present its proof separately (cf. Theorem 5.7 below).

Fix ample  $\mathbf{Q}$ -divisors  $A_1, \dots, A_r$  whose classes in  $N^1(X)_{\mathbf{Q}}$  form a basis. It is convenient to take on  $N^1(X)_{\mathbf{Q}}$  the norm  $\|\sum_{i=1}^r \alpha_i A_i\| = \max_i |\alpha_i|$ . For a positive rational number  $s$ , and for a  $\mathbf{Q}$ -divisor  $D_0$  such that  $V \not\subseteq \mathbf{B}_+(D_0 - s \sum_{i=1}^r A_i)$ , we denote by  $T(D_0, s)$  the set of divisor classes  $D_0 - (p_1 A_1 + \dots + p_r A_r)$ , where  $0 \leq p_i \leq s$  are rational numbers. It is enough to show that for every such box  $T(D_0, s)$  there is a constant  $C = C(D_0, s)$  such that

$$(10) \quad |\text{vol}_{X|V}(D_1) - \text{vol}_{X|V}(D_2)| \leq C \cdot \|D_1 - D_2\|,$$

for every  $D_1$  and  $D_2$  in  $T(D_0, s)$ .

Let  $H$  be a fixed ample  $\mathbf{Q}$ -divisor, and  $D$  in  $T(D_0, s)$ . If  $\epsilon_0$  is such that  $V \not\subseteq \mathbf{B}_+(D_0 - s \sum_i A_i - \epsilon_0 H)$  and if  $\epsilon \leq \epsilon_0$ , then

$$(11) \quad \text{vol}_{X|V}(D - \epsilon H) \geq \text{vol}_{X|V}((1 - \epsilon/\epsilon_0)D) = (1 - \epsilon/\epsilon_0)^d \text{vol}_{X|V}(D).$$

By the openness of the ample cone, there is a positive real number  $b$ , such that if  $E$  is a  $\mathbf{Q}$ -divisor with  $\|E\| \leq b$ , then  $H - E$  is ample. If  $A$  is a  $\mathbf{Q}$ -divisor such that

$\|A\| \leq b \cdot \epsilon_0$ , we have that  $\frac{\|A\|}{b}H - A$  is ample. Combined with (11), this shows that for every  $D \in T(D_0, s)$

$$(12) \quad \text{vol}_{X|V}(D - A) \geq \text{vol}_{X|V}\left(D - \frac{\|A\|}{b}H\right) \geq \left(1 - \frac{\|A\|}{b\epsilon_0}\right)^d \text{vol}_{X|V}(D).$$

As  $d \geq 1$ , we see that there is a constant  $C'$  such that for every  $D$  and  $A$  as above we have

$$(13) \quad \text{vol}_{X|V}(D) - \text{vol}_{X|V}(D - A) \leq C' \cdot \|A\| \cdot \text{vol}_{X|V}(D) \leq C' \cdot \text{vol}_{X|V}(D_0) \cdot \|A\|.$$

Suppose now that  $D \in T(D_0, s)$  and that  $A$  is an effective linear combination of the  $A_i$  such that  $D - A \in T(D_0, s)$ . If  $m \gg 0$  and if we apply (13) successively to  $D - \frac{i}{m}A$  and  $\frac{1}{m}A$  for  $0 \leq i \leq m - 1$ , we deduce

$$(14) \quad |\text{vol}_{X|V}(D) - \text{vol}_{X|V}(D - A)| \leq \frac{C}{2} \cdot \|A\|,$$

where  $C = 2C' \cdot \text{vol}_X(V, D_0)$ .

We finish the proof as in the case of the usual volume function (see [Laz], Theorem 2.2.44). Note that if  $D_1, D_2 \in T(D_0, s)$ , then we may write  $D_2 = D_1 + E - F$ , where  $E$  and  $F$  are effective linear combinations of the  $A_i$ . We apply (14) to get

$$\text{vol}_{X|V}(D_1) - \text{vol}_{X|V}(D_1 - F) \leq \frac{C}{2} \cdot \|F\|,$$

$$\text{vol}_{X|V}(D_2) - \text{vol}_{X|V}(D_1 - F) \leq \frac{C}{2} \cdot \|E\|.$$

Since  $\|D_1 - D_2\| = \max\{\|E\|, \|F\|\}$ , (10) follows from the triangle inequality.

The fact that  $\text{vol}_{X|V}(-)^{1/d}$  is concave on the set of  $\mathbf{R}$ -divisor classes  $D$  such that  $V \not\subseteq \mathbf{B}_+(D)$  follows by continuity from Corollary 2.16.  $\square$

**Remark 5.3.** Note that in (13) in the above proof, we may take  $A$  to be numerically trivial. This gives another way of seeing that if  $V \not\subseteq \mathbf{B}_+(D)$ , then the restricted volume  $\text{vol}_{X|V}(D)$  depends only on the numerical class of  $D$ .

**Remark 5.4.** We chose to give the above proof of Theorem 5.2 a) that shows that more generally, every homogeneous function defined on the rational points of an open convex cone containing the ample cone, and which is non-decreasing with respect to adding an ample class is locally Lipschitz continuous. Alternatively, the assertion in the theorem could be deduced from the more subtle concavity property of the restricted volume function, plus the following well-known fact: a homogeneous convex function defined on the rational points of a convex domain is Lipschitz around every point in the domain (in particular, it is locally uniformly continuous, and therefore it can be extended by continuity to the whole domain).

**Example 5.5.** It follows from Theorem 5.2 that if  $D$  is a nef  $\mathbf{R}$ -divisor and if  $V \not\subseteq \mathbf{B}_+(D)$  is a subvariety of dimension  $d \geq 1$ , then

$$(15) \quad \text{vol}_{X|V}(D) = (D^d \cdot V).$$

Indeed, if  $D$  is a  $\mathbf{Q}$ -divisor, then the assertion follows from Corollary 2.17, and the general case follows by continuity. Moreover, the second part of Theorem 5.2 and the continuity of the intersection form imply that the equality (15) still holds if  $V \subseteq \mathbf{B}_+(D)$  is an irreducible component, and in this case both numbers are zero (recall that each irreducible component of  $\mathbf{B}_+(D)$  has positive dimension by Proposition 1.1).

On the other hand, since  $D$  is nef, if  $V \subseteq X$  is a subvariety of dimension  $d \geq 1$ , we have  $D|_V$  big if and only if  $(D^d \cdot V) > 0$ . This gives the following description of  $\mathbf{B}_+(D)$ , which generalizes to the case of an  $\mathbf{R}$ -divisor the main result of [Na1].

**Corollary 5.6.** *If  $D$  is a nef  $\mathbf{R}$ -divisor, then  $\mathbf{B}_+(D)$  is the union  $\text{Null}(D)$  of those subvarieties  $V$  of  $X$  such that the restriction  $D|_V$  is not big.*

As mentioned above, the main result on the behavior of the restricted volume is part (b) of Theorem 5.2. We discuss it in what follows.

**Theorem 5.7.** *Let  $X$  be a smooth projective complex variety, and let  $D$  be an  $\mathbf{R}$ -divisor on  $X$ . If  $V$  is an irreducible component of  $\mathbf{B}_+(D)$ , then*

$$(16) \quad \lim_{D' \rightarrow D} \text{vol}_{X|V}(D') = 0,$$

where the limit is over  $\mathbf{Q}$ -divisors  $D'$  whose classes go to the class of  $D$ .

**Remark 5.8.** Using the definition of  $\text{vol}_{X|V}(-)$  on  $\text{Big}^V(X)_{\mathbf{R}}$  and the concavity of restricted volumes in Corollary 2.16, we see that if  $V$  is an irreducible component of  $\mathbf{B}_+(D)$ , then Theorem 5.7 gives

$$\lim_{\xi \rightarrow D} \text{vol}_{X|V}(\xi) = 0,$$

where the limit is over all  $\xi \in \text{Big}^V(X)_{\mathbf{R}}$  such that  $\xi$  goes to the class of  $D$ . Hence we get the assertion in part (b) of Theorem 5.2.

**Corollary 5.9.** *For any  $\mathbf{Q}$ -divisor  $D$ , the irreducible components of  $\mathbf{B}_+(D)$  are precisely the maximal (with respect to inclusion)  $V \subseteq X$  such that  $\text{vol}_{X|V}(D) = 0$ .*

*Proof.* If  $V$  is a component of  $\mathbf{B}_+(D)$ , Theorem 5.7 implies that  $\text{vol}_{X|V}(D) = 0$ . On the other hand, if  $V \not\subseteq \mathbf{B}_+(D)$ , then we clearly have  $\text{vol}_{X|V}(D) > 0$ .  $\square$

**Example 5.10** (Failure of Theorem 5.7 for non-components). We give an example of a subvariety  $V$  that is properly contained in an irreducible component of the augmented base locus and for which the conclusion of Theorem 5.7 is no longer true. Consider  $\pi : X \rightarrow \mathbf{P}^3$  to be the blow-up of  $\mathbf{P}^3$  along a line  $l$ , with exceptional divisor  $E \cong \mathbf{P}^1 \times \mathbf{P}^1$  (where  $\pi$  induces the projection onto the first component). The line bundle  $L := \pi^* \mathcal{O}_{\mathbf{P}^3}(1)$  is big and globally generated on  $X$ , and so by Corollary 5.6 we have  $\mathbf{B}_+(L) = \text{Null}(L) = E$ .

Consider now a smooth curve  $C$  of type  $(2, 1)$  in  $E$ . It is easy to see that for all  $m$ , the image of the map

$$H^0(X, mL) \longrightarrow H^0(C, mL|_C)$$

is isomorphic to the image of

$$H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(m)) \longrightarrow H^0(l, \mathcal{O}_l(m)),$$

so  $\text{vol}_{X|C}(L) = 1$ . In particular, it is nonzero. We also see that in this case  $\|L \cdot C\| \neq \text{vol}_{X|C}(L)$  (compare with Theorem 2.13). Indeed, since  $L$  is globally generated we have  $\|L \cdot C\| = (L \cdot C) = 2$ .

Note that  $\text{vol}_{X|C}(-)$  is not continuous at  $L$ . Indeed, let  $L_m := L - \frac{1}{m}E$  be a sequence of  $\mathbf{Q}$ -divisors converging to  $L$ . We see that  $L_m$  is ample for  $m$  large enough, which implies that  $\text{vol}_{X|C}(L_m) = (L_m \cdot C)$  by Example 2.3. An easy computation shows

$$\text{vol}_{X|C}(L_m) = 2 + \frac{1}{m} \longrightarrow 2 \neq 1 = \text{vol}_{X|C}(L).$$

**Remark 5.11** (Reduction to the case of positive perturbations). Note that if  $D$  and  $A$  are  $\mathbf{Q}$ -divisors and if  $A$  is ample, then  $\text{vol}_{X|V}(D) \leq \text{vol}_{X|V}(D+A)$ . It follows that in order to prove Theorem 5.7, it is enough to consider the limit over those  $\mathbf{Q}$ -divisor classes  $D'$  such that  $D' - D$  is ample. Indeed, suppose that  $A_1, \dots, A_r$  are ample  $\mathbf{Q}$ -divisors whose classes give a basis of  $N^1(X)_{\mathbf{R}}$ . We consider on  $N^1(X)_{\mathbf{R}}$  the norm given by

$$\left\| \sum_i \alpha_i A_i \right\| := \max_i |\alpha_i|.$$

It follows that given a  $\mathbf{Q}$ -divisor  $D' \not\equiv D$ , there exists a  $\mathbf{Q}$ -divisor  $D''$  such that  $\|D'' - D\| = \|D' - D\|$  and such that both  $(D'' - D)$  and  $(D'' - D')$  lie in the convex cone spanned by the  $A_i$ . To see this, if  $D' - D = \sum_i \beta_i A_i$ , simply take  $D'' - D = \sum_i |\beta_i| A_i$ .

**Proof of Theorem 5.7.** We start with two lemmas. First, let  $X$  be a smooth projective variety of dimension  $n$ , and let  $V \subset X$  be an irreducible subvariety of dimension  $d$ . We recall the definition of the asymptotic order function  $\text{ord}_V(\|\cdot\|)$  defined on  $\text{Big}(X)_{\mathbf{R}}$  (we refer to [ELMNP2] for the basic properties of this function). If  $D$  is a big  $\mathbf{Q}$ -divisor, and if  $m$  is divisible enough, then  $\text{ord}_V(|mD|)$  denotes the order of vanishing at the generic point of  $V$  of a general element in  $|mD|$ . We have

$$\text{ord}_V(\|D\|) := \lim_{m \rightarrow \infty} \frac{\text{ord}_V(|mD|)}{m} = \inf_m \frac{\text{ord}_V(|mD|)}{m}.$$

This extends as a continuous, convex function to  $\text{Big}(X)_{\mathbf{R}}$ . Note that the notation in the asymptotic order of vanishing of  $D$  should not be confused with the norm on  $N^1(X)_{\mathbf{R}}$ . For the rest of this section, we fix as above a basis  $A_1, \dots, A_r$  for  $N^1(X)_{\mathbf{R}}$  consisting of ample divisors, and let  $\sigma$  be the cone generated by the  $A_i$ . We consider the norm on  $N^1(X)_{\mathbf{R}}$  given by  $\left\| \sum \alpha_i A_i \right\| = \max_i |\alpha_i|$ .

**Lemma 5.12.** *Given  $\sigma$  as above, there is  $\beta > 0$  such that for every big  $\mathbf{R}$ -divisor  $D$  and for every  $V \subseteq \mathbf{B}_+(D)$ , if  $A \in \sigma$  is nonzero and  $D - A$  is big, we have*

$$(17) \quad \text{ord}_V(\|D - A\|) \geq \beta \cdot \|A\|.$$

*Proof.* We adapt to our more general setting the argument for Lemma 1.4 in [Na2]. It is clear by the continuity of the asymptotic order function that it is enough to satisfy the condition in the statement for every nonzero  $A \in \sigma$  such that  $D - A$  is a big  $\mathbf{Q}$ -divisor.

Let  $A'$  be a very ample divisor on  $X$  such that  $T_X \otimes \mathcal{O}(A')$  is an ample vector bundle. We can find a positive integer  $b$  such that for every nonzero  $A \in \sigma$ , the divisor  $\frac{b}{\|A\|} \cdot A - A'$

is ample. We show now that  $\beta = \frac{1}{b}$  satisfies our requirement. Suppose that there is a nonzero  $A \in \sigma$  such that  $D - A$  is a big  $\mathbf{Q}$ -divisor and  $\text{ord}_V(\|D - A\|) < \beta \cdot \|A\|$ . This means that there is  $m$  and  $s \in H^0(X, \mathcal{O}(m(D - A)))$  whose order of vanishing at the generic point  $\eta$  of  $V$  is  $\text{ord}_\eta(s) < m\beta \cdot \|A\|$ . Note that we may replace  $m$  by any multiple, so we may assume that  $m$  is divisible enough.

We use the notation in [ELN]: if  $B$  is a line bundle on  $X$ , then  $\mathcal{D}_B^\ell$  denotes the *bundle of differential operators* of order  $\leq \ell$  on  $B$ . This is defined as

$$\mathcal{D}_B^\ell := \mathcal{H}om(P_B^\ell, B) = P_B^{\ell \vee} \otimes B,$$

where  $P_B^\ell$  is the bundle of  $\ell$ -principal parts associated to  $B$  (having as fibers the spaces of  $\ell$ -jets of sections of  $B$ ). These bundles sit in short exact sequences of the form

$$0 \longrightarrow \mathcal{D}_B^{\ell-1} \longrightarrow \mathcal{D}_B^\ell \longrightarrow \text{Sym}^\ell(T_X) \longrightarrow 0.$$

It follows from Lemma 2.5 in [ELN] and our hypothesis on  $A'$  that there is  $\ell_0$  such that for every  $B$  and every  $\ell \geq \ell_0$ , the sheaf  $\mathcal{D}_B^\ell \otimes \mathcal{O}_X(\ell A')$  is globally generated. Since every global section of  $B$  determines a vector bundle map  $\mathcal{D}_B^\ell \rightarrow B$ , we have a natural induced map

$$H^0(\mathcal{D}_B^\ell \otimes \mathcal{O}_X(\ell A')) \longrightarrow H^0(\mathcal{O}_X(B + \ell A'))$$

which eventually produces nontrivial sections in  $H^0(\mathcal{O}_X(B + \ell A'))$  arising locally via the process of differentiation.

For our Lemma, by taking  $B = \mathcal{O}(m(D - A))$  it follows that if  $\ell \geq m\beta \cdot \|A\|$ , then we may apply a suitable differential operator to our section  $s$  to get  $\tilde{s} \in H^0(X, \mathcal{O}(m(D - A) + \ell A'))$  that does not vanish at  $\eta$ . If  $mA - \ell A'$  is ample, then this contradicts the assumption that  $V$  is contained in  $\mathbf{B}_+(D)$ .

As  $\frac{1}{\beta \cdot \|A\|} \cdot A - A'$  is ample by assumption, we may take  $m$  large enough so that

$$\frac{m}{m\beta \cdot \|A\| + 1} \cdot A - A'$$

is again ample. Moreover, we may assume that  $m\beta \cdot \|A\| \geq \ell_0 + 1$ . If we choose  $\ell$  to be the smallest integer  $\geq m\beta \cdot \|A\|$ , then both our requirements on  $\ell$  are satisfied. This completes the proof of the lemma.  $\square$

Our next lemma deals with a subtracting procedure introduced in [Na2] (note however that we add an extra condition in order to fix a small gap in the proof in *loc. cit.*). The goal is to get a lower bound on the dimension of the space of sections of  $L$  minus an ample, starting from sections in  $L$  with small order of vanishing at given points.

We keep the assumption that  $V$  is a subvariety of dimension  $d \geq 1$  of the smooth, projective variety  $X$ , and let  $L$  and  $B$  be  $\mathbf{Q}$ -divisors on  $X$  such that  $B$  and  $B - L$  are ample. We consider integers  $m$  and  $k$  such that  $m$  is divisible enough and  $k \gg m$ . Suppose that we have positive rational numbers  $\epsilon < \epsilon'$  such that if  $x_1, \dots, x_N \in V$  are general points, for every  $m$  (divisible enough)  $mL$  separates  $m\epsilon'$  jets at  $x_1, \dots, x_N$ . Suppose that for every  $m$  and  $k$  we have a closed subscheme  $V_{m,k}$  of  $X$  supported on  $V$ . If  $I_V$  and  $I_{m,k}$  are the ideals defining  $V$  and  $V_{m,k}$ , respectively, we assume that around every smooth point of  $V$  we have  $I_{m,k} \subseteq I_V^{km\epsilon}$ .

Let us fix now  $x_1, \dots, x_N$  smooth points on  $V$  as above such that  $\text{depth}(\mathcal{O}_{V_{m,k},x_i}) = d$  for all  $i$  (this assumption is satisfied by general points). We fix also a positive rational number  $a$ . Let  $m_1$  be divisible enough such that  $m_1 a B$  is very ample and the linear system

$$\Sigma = \{B_0 \in |m_1 a B| \mid x_i \in B_0\}$$

induces a basepoint-free linear system on  $X \setminus \{x_1, \dots, x_N\}$ . If  $s$  is a section of a line bundle on a scheme  $Z$ , the order of vanishing of  $s$  at a point  $x \in Z$  is the largest  $p$  such that a local equation for  $s$  at  $x$  lies in the  $p$ th power of the ideal defining  $x$  in  $Z$ .

**Lemma 5.13.** *With the above notation, suppose that for every  $m$  and  $k$  we have a vector subspace  $W_{m,k} \subseteq H^0(X, \mathcal{O}(kmL))$  such that*

- (i) *For every nonzero section  $s$  in  $W_{m,k}$  we have  $\min_i \text{ord}_{x_i}(s) < km\epsilon$ .*
- (ii) *If  $B'$  is a very general divisor in  $\Sigma$ , then for every  $m$  and  $k$ , and every  $s$  in  $W_{m,k}$  such that  $s|_{B'}$  is nonzero, we have  $\min_i \text{ord}_{x_i}(s|_{B'}) < km\epsilon$ .*

*Then  $W_{m,k}$  induces a vector subspace  $W'_{m,k} \subseteq H^0(V_{m,k}, \mathcal{O}(km(L - aB)))$  such that every nonzero section  $s$  of  $W'_{m,k}$  satisfies*

$$(18) \quad \min_i \text{ord}_{x_i}(s) < km\epsilon$$

*and such that*

$$\dim W'_{m,k} \geq \dim W_{m,k} - a \cdot \deg_B(V) \cdot \ell(\mathcal{O}_{V_{m,k},\eta}) \cdot (km)^d,$$

*where  $\eta$  denotes the generic point of  $V$ .*

*Proof.* Note first that since around every  $x_i$  we have  $I_{m,k} \subseteq I_V^{km\epsilon}$ , a section of a line bundle on a subscheme  $Z$  of  $X$  has order  $< km\epsilon$  at  $x_i$  if and only if this holds for its restriction to  $Z \cap V_{m,k}$ . We assume that  $m$  is divisible by  $m_1$ , and let  $B_1, \dots, B_{km/m_1}$  be very general elements in the linear system  $\Sigma$ , so they satisfy the condition (ii) above. Since  $\text{depth}(\mathcal{O}_{V_{m,k},x_i}) \geq 1$ , we may also assume that no  $B_i$  contains an associated point of  $V_{m,k}$ .

Let  $\overline{W}_{m,k}$  denote the image of  $W_{m,k}$  in  $H^0(V_{m,k}, \mathcal{O}(kmL))$ . Our assumption implies that the restriction map gives an isomorphism  $W_{m,k} \simeq \overline{W}_{m,k}$ . Let  $W'_{m,k}$  be the kernel of the composition

$$\overline{W}_{m,k} \hookrightarrow H^0(V_{m,k}, \mathcal{O}(kmL)) \rightarrow H^0(V_{m,k}, \mathcal{O}_{V_{m,k}}(kmL)|_{\sum_i B_i}).$$

It is clear that  $W'_{m,k} \subseteq H^0(V_{m,k}, \mathcal{O}(km(L - aB)))$  and that (i) above implies that the nonzero sections in  $W'_{m,k}$  satisfy (18).

In order to get the lower bound for  $\dim W'_{m,k}$ , it is enough to show that if

$$w_p := \text{Im}(W_{m,k} \longrightarrow H^0(V_{m,k} \cap B_p, \mathcal{O}_{V_{m,k} \cap B_p}(kmL))),$$

then for every  $p \leq km/m_1$  we have the following upper bound:

$$\dim(w_p) \leq m_1 a \cdot \deg_B(V) \cdot \ell(\mathcal{O}_{V_{m,k},\eta}) \cdot (km)^{d-1}.$$

Since  $B - L$  is ample, it follows that  $mB$  also separates  $m\epsilon'$ -jets at  $x_1, \dots, x_N$ , hence we can find  $m_2$  such that if  $D_1, \dots, D_{d-1}$  are general elements in  $|m_2 B|$  with order of

vanishing  $\geq m_2\epsilon + 1$  at each  $x_j$ , then their local equations form a regular sequence with respect to  $V_{m,k} \cap B_p$ . We use here the fact that  $\text{depth}(\mathcal{O}_{V_{m,k} \cap B_p, x_j}) = d - 1$  for all  $j$  and apply Remark 4.4 successively to avoid containing suitable associated points.

From now on we assume that  $m$  is divisible also by  $m_2$ . Property (ii) above implies that no nonzero element in  $w_i$  can lie in the image of

$$H^0\left(V_{m,k} \cap B_p, \mathcal{O}\left(km\left(L - \frac{1}{m_2} \cdot D_1\right)\right)\right) \longrightarrow H^0(V_{m,k} \cap B_p, \mathcal{O}(kmL)).$$

Therefore  $\dim(w_p)$  is bounded above by the dimension of the image of  $W_{m,k}$  in  $H^0(V_{m,k} \cap B_p \cap km/m_2 \cdot D_1, \mathcal{O}(kmL))$ . Moreover, since  $\frac{km}{m_2}D_1$  passes through  $x_1, \dots, x_N$  with multiplicity at least  $km\epsilon$ , it follows that for every  $s$  in  $W_{m,k}$  such that  $s|_{B_p}$  is nonzero, we have

$$\min_j \text{ord}_{x_j}(s|_{B_p \cap D_1}) < km\epsilon.$$

Therefore we can repeat the above procedure. After  $d - 1$  such steps, we deduce

$$\dim(w_p) \leq m_1 a \cdot (km)^{d-1} \cdot \deg_B(V_{m,k}) = m_1 a \cdot (km)^{d-1} \cdot \deg_B(V) \cdot \ell(\mathcal{O}_{V_{m,k}, \eta}),$$

which completes the proof of the lemma.  $\square$

**Remark 5.14.** Suppose that in the above lemma we assume that all sections in  $W_{m,k}$  restrict to zero on subschemes  $V'_{m,k}$  whose support is properly contained in  $V$ . If the points  $x_1, \dots, x_N$  do not lie in any support of a primary component of a  $V'_{m,k}$  (which can be achieved by taking the  $x_i$  very general on  $V$ ), then we get that the sections in  $W'_{m,k}$  also restrict to zero on  $V'_{m,k}$ . Indeed, in the above proof it is enough to make sure that the divisors  $B_1, \dots, B_{km/m_1}$  do not contain any of the supports of the primary components of the schemes  $V'_{m,k}$ .

We can give now the proof of Theorem 5.7. We will use the following notation: if  $E$  is a divisor on  $X$  such that  $|E| \neq \emptyset$ , we will denote by  $\mathfrak{b}_{|E|}$  the ideal defining the base locus of this linear system.

*Proof of Theorem 5.7.* The proof of Theorem 5.7 follows the approach in [Na2], using in addition our result on approximating volumes in terms of separation of jets. Note that by Proposition 1.1, we have  $\dim(V) = d \geq 1$ . We may also assume that  $D$  is big: otherwise  $V = X$ , and the theorem follows from the continuity of the usual volume function on  $N^1(X)_{\mathbf{R}}$  (see [Laz], Corollary 2.2.45). While the proof of the general case is quite technical, if we assume that  $V = \mathbf{B}_+(D)$ , then the proof becomes more transparent. For the benefit of the reader, we give first the proof of this particular case, and we describe later the general argument.

We start therefore by assuming that  $V = \mathbf{B}_+(D)$  and that  $D$  contradicts the conclusion of the Theorem. It follows from Remark 5.11 that there is  $\delta > 0$  and a sequence of ample divisors  $A_q$  going to 0 and such that  $D + A_q$  are  $\mathbf{Q}$ -divisors with  $\text{vol}_{X|V}(D + A_q) > \delta$  for every  $q$ . Moreover, we may clearly assume that  $V \not\subseteq \mathbf{B}_+(D + A_q)$  for all  $q$ .

By Lemma 5.12, we can find  $\beta \in \mathbf{Q}_+^*$  such that  $\text{ord}_V(\|D - H\|) \geq \beta \cdot \|H\|$  for every  $H$  in the interior of our cone  $\sigma$  such that  $D - H$  is big. In addition to this



lower bound for the asymptotic order function, we will also need an upper bound for the asymptotic multiplicity. Recall from [ELMNP2] that if  $E$  is a big  $\mathbf{Q}$ -divisor such that  $V$  is not properly contained in  $\mathbf{B}(E)$ , and if for  $m$  divisible enough we denote by  $e_m$  the Samuel multiplicity of the local ring  $\mathcal{O}_{X,V}$  with respect to the localization of  $\mathfrak{b}_{|mE|}$ , then this asymptotic multiplicity is defined by

$$e_V(\|E\|) := \lim_{m \rightarrow \infty} \frac{e_m}{m^{n-d}} = \inf_m \frac{e_m}{m^{n-d}}.$$

In our setup, since  $V \not\subseteq \mathbf{B}_+(D + A_q)$  for any  $q$ , it follows that  $e_V(\|D\|) = 0$ . Moreover,  $e_V(\|\cdot\|)^{1/(n-d)}$  is locally Lipschitz continuous, hence there is  $M > 0$  such that

$$(19) \quad e_V(\|D - H\|) < M \cdot \|H\|^{n-d}$$

if  $H$  lies in a suitable ball  $\mathcal{U}$  around the origin. We refer to [ELMNP2], the end of §2 and Remark 3.2 for the properties of asymptotic multiplicity that we used (the fact that  $e_V(\|\cdot\|)^{1/(n-d)}$  is locally Lipschitz continuous on its domain follows also from the fact that it is homogeneous and convex).

We choose now a  $\mathbf{Q}$ -divisor  $B$  such that both  $B$  and  $B - D$  lie in the interior of the cone  $\sigma$ . Let  $a \in \mathbf{Q}_+^*$  be small enough, such that  $\mathbf{B}_+(D - aB) = \mathbf{B}_+(D)$  and

$$(20) \quad a < \frac{(n-d)! \cdot \delta \beta^{n-d}}{n! \cdot M \cdot \deg_B(V)}.$$

We fix now  $q$  such that  $aB - A_q$  and  $B - (D + A_q)$  lie in the interior of  $\sigma$  and from now on we put  $A = A_q$ .

Since  $V \not\subseteq \mathbf{B}_+(D + A)$  and  $\text{vol}_{X|V}(D + A) > \delta$ , Theorem 4.2 implies that there are  $N \geq 1$  and  $\epsilon \in \mathbf{Q}_+^*$  such that  $N\epsilon^d > \delta$  and if  $x_1, \dots, x_N$  are general points on  $V$ , the canonical map

$$(21) \quad H^0(X, \mathcal{O}(p(D + A))) \rightarrow \bigoplus_{i=1}^N H^0(X, \mathcal{O}(p(D + A)) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{p\epsilon})$$

is surjective for  $p$  divisible enough. Moreover, by taking  $N$  large enough we can make sure that  $\epsilon$  is as small as we want.

We fix now  $H$  small enough in the interior of  $\sigma \cap \mathcal{U}$  such that  $aB - A - H$  is an ample  $\mathbf{Q}$ -divisor. Let us choose  $\epsilon$  and  $N$  as above such that  $\epsilon/\beta < \|H\|$ . After subtracting from  $H$  a small multiple of a  $\mathbf{Q}$ -divisor, we can make  $\|H\|$  arbitrarily close to  $\epsilon/\beta$ , and therefore by (20) we may assume that

$$(22) \quad a < \frac{(\epsilon/\beta)^{n-d}}{\|H\|^{n-d}} \cdot \frac{(n-d)! \cdot \delta \beta^{n-d}}{n! \cdot M \cdot \deg_B(V)}.$$

We see that  $D - H$  is a  $\mathbf{Q}$ -divisor and  $\mathbf{B}_+(D) \subseteq \mathbf{B}(D - H) \subseteq \mathbf{B}_+(D - aB)$ , hence  $\mathbf{B}(D - H) = V$ . Note that we may assume in addition that (21) is surjective also for some  $\epsilon' > \epsilon$ , if  $p$  is divisible enough.

We will consider integers  $m$  and  $k$ , with  $m$  large and divisible enough, and with  $k \gg m$ . For every such  $k$  and  $m$ , let  $\pi_m: X_m \rightarrow X$  be a log resolution of  $\mathfrak{b}_{|m(D-H)|}$ , and write  $(\pi_m)^{-1}(|m(D-H)|) = E_m + |M_m|$ , with  $E_m$  the fixed part and  $M_m$  the moving

part. The ideal  $I_{m,k} = (\pi_m)_* \mathcal{O}_{X_m}(-kE_m)$  is equal to the integral closure of  $\mathfrak{b}_{|m(D-H)|}^k$  and it defines a subscheme that we denote by  $V_{m,k}$ . Note that since  $\mathbf{B}(D-H) = V$  and  $m$  is divisible enough,  $V_{m,k}$  is supported on  $V$ . We denote by  $I_V$  the ideal defining  $V$ . By our choice of  $\beta$ , and since  $\epsilon/\beta < \|H\|$ , we see that in the neighborhood of any smooth point of  $V$  we have

$$(23) \quad I_{m,k} \subseteq I_V^{km\epsilon}$$

(we use the fact that for an ideal defining a smooth subvariety, the symbolic powers coincide with the usual powers and they are integrally closed).

Let  $m_1$  be divisible enough, so  $m_1 aB$  is very ample, the linear system

$$\Sigma = \{B_0 \in |m_1 aB| \mid x_i \in B_0 \text{ for all } i\}$$

has no base points in  $X \setminus \{x_1, \dots, x_N\}$ , and a general element of  $\Sigma$  is smooth. We choose  $N$  general points  $x_1, \dots, x_N$  such that the above properties of  $\Sigma$  are satisfied, and in addition (21) is surjective and  $\text{depth}(\mathcal{O}_{V_{m,k}, x_i}) = d$  for every  $i$ . Moreover, we can choose these points such that if  $p$  is divisible enough, then  $p(D+A) - m_1 aB$  separates  $p\epsilon$ -jets at  $x_1, \dots, x_N$  (see Remark 4.6).

Our plan now is to apply Lemma 5.13. To this end, fix any very general divisor  $B_0 \in |m_1 aB|$  passing through  $x_1, \dots, x_N$ . We have the following commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}(km(D+A) - m_1 aB)) & \xrightarrow{\alpha_{m,k}} & H^0(X, \mathcal{O}(km(D+A))) \\ \downarrow \phi_{m,k} & & \downarrow \psi_{m,k} \\ \oplus_{i=1}^N H^0(\mathcal{O}(km(D+A) - m_1 aB) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{km\epsilon}) & \xrightarrow{\beta_{m,k}} & \oplus_{i=1}^N H^0(\mathcal{O}(km(D+A)) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{km\epsilon}) \end{array}$$

where the horizontal maps are induced by local equations of  $B_0$ . Note that  $\alpha_{m,k}$  is injective, and by construction,  $\phi_{m,k}$  and  $\psi_{m,k}$  are surjective. Therefore we can choose  $W_{m,k} \subseteq H^0(X, \mathcal{O}(km(D+A)))$  such that  $W_{m,k}$  is mapped isomorphically by  $\psi_{m,k}$  onto  $\oplus_{i=0}^N H^0(X, \mathcal{O}(km(D+A)) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{km\epsilon})$  and such that  $\text{Im}(\beta_{m,k})$  is in the image of  $W_{m,k} \cap \text{Im}(\alpha_{m,k})$ . This implies that for every nonzero section  $s$  in  $W_{m,k}$ , we have

$$(24) \quad \min_i \text{ord}_{x_i}(s) < km\epsilon.$$

Moreover, if  $s|_{B_0}$  is nonzero, then

$$(25) \quad \min_i \text{ord}_{x_i}(s|_{B_0}) < km\epsilon.$$

We assert moreover that the analogue of (25) holds for any very general  $B'_0$  in  $|m_1 aB|$  passing through the  $x_i$ . Indeed, consider for such  $B'_0$  the commutative diagram

$$\begin{array}{ccc} W_{m,k} & \longrightarrow & H^0(B'_0, \mathcal{O}(km(D+A))) \\ \downarrow \psi_{m,k} & & \downarrow \rho_{m,k}(B'_0) \\ J_{m,k} & \longrightarrow & \overline{J}_{m,k}(B'_0) \end{array}$$

where  $J_{m,k} = \bigoplus_{i=1}^N H^0(X, \mathcal{O}(km(D+A)) \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^{km\epsilon})$  and

$$\bar{J}_{m,k}(B'_0) = \bigoplus_{i=1}^N H^0(B'_0, \mathcal{O}(km(D+A)) \otimes \mathcal{O}_{B'_0,x_i}/\mathfrak{m}_{x_i}^{km\epsilon}).$$

Since  $\psi_{m,k}$  is surjective, we see that the restriction of  $\rho_{m,k}(B'_0)$  to the image of  $W_{m,k}$  in  $H^0(B'_0, \mathcal{O}(km(D+A)))$  is always surjective. Our assertion is that this image maps isomorphically onto  $\bar{J}_{m,k}(B'_0)$ . By construction, this holds when  $B'_0 = B_0$ , and since the spaces involved have constant dimension for general  $B'_0$ , it follows that (25) holds with  $B_0$  replaced by  $B'_0$ . Therefore the two hypotheses (i) and (ii) in Lemma 5.13 are satisfied for  $L = D + A$ .

By construction we have

$$(26) \quad \dim W_{m,k} = \sum_{i=1}^N \dim \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^{km\epsilon} = N \binom{km\epsilon + n - 1}{n}.$$

Since  $N\epsilon^d > \delta$ , we deduce

$$(27) \quad \dim W_{m,k} > \frac{\delta \epsilon^{n-d}}{n!} (km)^n + O((km)^{n-1}).$$

On the other hand, Lemma 5.13 gives a vector subspace

$$W'_{m,k} \subseteq H^0(V_{m,k}, \mathcal{O}(km(D+A-aB)))$$

such that

$$(28) \quad \dim W'_{m,k} \geq \dim W_{m,k} - a \cdot \deg_B(V) \cdot \ell(\mathcal{O}_{V_{m,k},\eta}) \cdot (km)^d$$

and such that for every nonzero section  $s$  in  $W'_{m,k}$ , we have

$$(29) \quad \min_{i=1}^N \text{ord}_{x_i}(s) < km\epsilon.$$

Since  $aB - A - H$  is ample, we get corresponding spaces of sections

$$W''_{m,k} \subseteq H^0(V_{m,k}, \mathcal{O}(km(D-H)))$$

satisfying the same lower bound on the dimension and such that for every nonzero section in  $W''_{m,k}$  we have (29).

We give now an upper bound for  $\ell(\mathcal{O}_{V_{m,k},\eta})$  when  $m$  is divisible enough, but fixed, and  $k$  goes to infinity. We clearly have

$$(30) \quad \ell(\mathcal{O}_{V_{m,k},\eta}) \leq \ell(\mathcal{O}_{X,\eta}/\mathfrak{b}_{|m(D-H)|}^k) < \frac{\tilde{e}_m}{(n-d)!} \cdot k^{n-d}$$

if  $\tilde{e}_m$  is larger than the multiplicity  $e_m$  of  $\mathcal{O}_{X,\eta}$  with respect to the localization of  $\mathfrak{b}_{|m(D-H)|}$ . By (19), since  $m$  is large enough we may choose such  $\tilde{e}_m$  with  $\tilde{e}_m < M \cdot m^{n-d} \|H\|^{n-d}$  and conclude that

$$(31) \quad \ell(\mathcal{O}_{V_{m,k},\eta}) < \frac{M \cdot \|H\|^{n-d}}{(n-d)!} (km)^{n-d} \text{ for } k \gg 0.$$

Combining this with (28), (27) and (22) we deduce that if  $m$  is large and divisible enough, then  $\dim W''_{m,k}$  grows like a polynomial of degree  $n$  in  $k$ , when  $k$  goes to infinity.

We use this to construct global sections of  $km(D - H)$ . Consider the exact sequence  $H^0(X, \mathcal{O}(km(D - H))) \longrightarrow H^0(V_{m,k}, \mathcal{O}(km(D - H))) \longrightarrow H^1(X, \mathcal{O}(km(D - H)) \otimes I_{m,k})$ . Since  $M_m$  is nef, we have  $h^1(X_m, \mathcal{O}(kM_m)) \leq O(k^{n-1})$  for  $k \gg 0$ . Using the Leray spectral sequence, this gives

$$h^1(X, \mathcal{O}_X(km(D - H)) \otimes I_{m,k}) \leq O(k^{n-1})$$

for  $k \gg 0$ . Therefore most of the sections in  $W''_{m,k}$  can be lifted to  $H^0(X, \mathcal{O}(km(D - H)))$ .

On the other hand, recall that for every nonzero section  $s$  in  $W''_{m,k}$  we can find a point  $x_i$  such that  $\text{ord}_{x_i}(s) < km\epsilon$ . Since  $\text{ord}_V(|km(D - H)|) \geq km\beta \parallel H \parallel$  and  $\epsilon/\beta < \parallel H \parallel$ , we get a contradiction. This completes the proof in the case  $\mathbf{B}_+(D) = V$ .

We treat now the general case. The above proof fails since the subschemes  $V_{m,k}$  as defined above are not supported on  $V$  anymore. We need to do some extra work to ensure that the sections we construct on  $V_{m,k}$  can be extended to subschemes supported on the whole  $\mathbf{B}_+(D)$ . We will use the following notation: if  $F$  is an integral divisor such that  $|F| \neq \emptyset$ , then we denote the integral closure of  $\mathfrak{b}_{|F|}^k$  by  $\mathfrak{b}_{|F|}^{(k)}$ . We consider also  $\mathfrak{b}_{|F|}^{(k)2}$  that is defined locally as the ideal of sections  $\phi$  in  $\mathcal{O}_X$  such that for every divisor  $T$  over  $X$ , with center on  $X$  different from  $V$ , we have  $\text{ord}_T(\phi) \geq k \cdot \text{ord}_T(\mathfrak{b}_{|F|})$ . Note that for every  $F_1$  and  $F_2$ , it follows from definition that

$$(32) \quad \mathfrak{b}_{|mF_1|}^{(k)2} \cdot \mathfrak{b}_{|mF_2|}^{(k)2} \subseteq \mathfrak{b}_{|mF_1+mF_2|}^{(k)2}.$$

It is clear that if  $V$  is not contained in  $\mathbf{B}(F)$ , then  $\mathfrak{b}_{|mF|}^{(k)} = \mathfrak{b}_{|mF|}^{(k)2}$  for  $m$  divisible enough. On the other hand, if  $V$  is an irreducible component of  $\mathbf{B}(F)$  and  $m$  is divisible enough, then  $\mathfrak{b}_{|mF|}^{(k)}$  has a uniquely determined primary component supported on  $V$ . If this is defined by  $\mathfrak{b}_{|mF|}^{(k)1}$ , then we have  $\mathfrak{b}_{|mF|}^{(k)} = \mathfrak{b}_{|mF|}^{(k)1} \cap \mathfrak{b}_{|mF|}^{(k)2}$ .

We assume for the moment the following technical lemma.

**Lemma 5.15.** *Fix an ample divisor  $\overline{B}$  such that  $D - \overline{B}$  is a  $\mathbf{Q}$ -divisor with  $\mathbf{B}(D - \overline{B}) = \mathbf{B}_+(D)$ . There is a sheaf of ideals  $\mathcal{I}$  on  $X$  whose support does not contain  $V$  such that*

$$\mathcal{I}^{kp} \cdot \mathfrak{b}_{|p\gamma(D+A)|}^{(k)} \subseteq \mathfrak{b}_{|p(\gamma+1)(D+A-\frac{1}{\gamma}B_2)|}^{(k)2}$$

for every  $A$ ,  $B_2$ ,  $\gamma$ ,  $p$  and  $k$  as follows:  $A$  is ample such that  $D + A$  is a  $\mathbf{Q}$ -divisor with  $V \not\subseteq \mathbf{B}(D + A)$ ,  $B_2$  is an ample  $\mathbf{Q}$ -divisor such that  $\overline{B} - 2B_2$  is ample,  $\gamma > 1$  a rational number,  $p$  is divisible enough (depending on  $A$ ,  $B_2$ , and  $\gamma$ ), and  $k$  is an arbitrary positive integer.

In order to prove the general case of the theorem, we start by fixing  $\delta$ ,  $\beta$  and  $M$  as before. We fix also  $\overline{B}$  and  $\mathcal{I}$  as in the above lemma. Let  $B_1$  be an integral divisor such that both  $B_1$  and  $B_1 - D$  lie in the interior of  $\sigma$  and  $\mathcal{I} \otimes \mathcal{O}(B_1)$  is globally generated. Let  $B_2$  be a divisor in the interior of  $\sigma$  such that  $\overline{B} - 2B_2$  is ample and we put  $B = B_1 + B_2$ .

We choose now  $a \in \mathbf{Q}_+^*$  small enough such that  $\mathbf{B}_+(D - aB) = \mathbf{B}_+(D)$ ,  $a < 1$  and

$$(33) \quad a(1+a)^d < \frac{(n-d)! \cdot \delta \beta^{n-d}}{n! \cdot M \cdot \deg_B(V)}.$$

As before, we can choose  $A = A_q$  for  $q \gg 0$  such that  $aB_2 - A$  lies in the interior of  $\sigma$ . Moreover, we can choose  $\epsilon \in \mathbf{Q}_+^*$  and  $N \geq 1$  such that  $N\epsilon^d > \delta$  and (21) is surjective for very general points  $x_1, \dots, x_N$  on  $V$  if  $p$  is divisible enough (we may assume that the same map is surjective also for some  $\epsilon' > \epsilon$ ). Arguing as before, we can find a divisor  $H$  in the interior of  $\sigma \cap \mathcal{U}$  such that  $aB_2 - A - H$  is an ample  $\mathbf{Q}$ -divisor and we have  $\epsilon/\beta < \|H\|$  and

$$(34) \quad a(a+1)^d < \frac{(\epsilon/\beta)^{n-d}}{\|H\|^{n-d}} \cdot \frac{(n-d)! \cdot \delta \beta^{n-d}}{n! \cdot M \cdot \deg_B(V)}.$$

Note that  $D - H$  is a  $\mathbf{Q}$ -divisor with  $\mathbf{B}(D - H) = \mathbf{B}_+(D)$ . Let  $V_{m,k}$  be the subscheme defined by the primary ideal  $\mathfrak{b}_{|m(D-H)|}^{(k)_1}$ , so  $V_{m,k}$  has support  $V$ . We choose again  $m_1$  divisible enough, and very general points  $x_1, \dots, x_N$  on  $V$  such that  $m_1 aB$  is very ample, the linear subsystem of  $|m_1 aB|$  passing through  $x_1, \dots, x_N$  has no base points in  $X \setminus \{x_1, \dots, x_N\}$  and a general element is smooth, (21) is surjective and  $\text{depth}(\mathcal{O}_{V_{m,k}, x_i}) = d$  for every  $i$ . Moreover, if  $p$  is divisible enough, then  $p(D + A) - m_1 aB$  separates  $p\epsilon$ -jets at  $x_1, \dots, x_N$ . We may assume also that the points  $x_i$  do not lie on the support of any irreducible component of the schemes defined by the ideals  $\mathfrak{b}_{|m(D+A)|}^{(k)}$ .

We claim that we can find vector subspaces

$$\widetilde{W}_{m,k} \subseteq H^0\left(X, \mathcal{O}(km(a+1)(D+A)) \otimes \mathfrak{b}_{|m(D+A)|}^{(k)}\right)$$

such that the induced map

$$(35) \quad \widetilde{W}_{m,k} \rightarrow \bigoplus_{i=1}^N H^0(X, \mathcal{O}(km(a+1)(D+A)) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{km\epsilon+1})$$

is an isomorphism and for a very general  $B_0 \in |m_1 aB|$  passing through  $x_1, \dots, x_N$  and for every  $s \in \widetilde{W}_{m,k}$  with  $s|_{B_0} \neq 0$ , there is a point  $x_i$  such that  $\text{ord}_{x_i}(s|_{B_0}) \leq km\epsilon$ . Indeed, suppose first that  $B_0$  is very general as above, but fixed. Arguing as before, we see that we can find  $W_{m,1} \subseteq H^0(X, \mathcal{O}(m(D+A)))$  that maps isomorphically onto

$$\bigoplus_{i=1}^N H^0(X, \mathcal{O}(m(D+A)) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{m\epsilon})$$

and a subspace  $W_{m,1}^\circ \subseteq W_{m,1}$  that maps isomorphically onto

$$\bigoplus_{i=1}^N H^0(X, \mathcal{O}(m(D+A) - m_1 aB_0) \otimes \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^{m\epsilon}).$$

Note also that we have  $H^0(X, \mathcal{O}(m(D+A))) = H^0(X, \mathcal{O}(m(D+A)) \otimes \mathfrak{b}_{|m(D+A)|})$ .

We can choose now subspaces  $W_{m,k}^\circ \subseteq W_{m,k} \subseteq H^0(X, \mathcal{O}(km(D+A)))$  such that  $W_{m,k}$  maps isomorphically onto

$$\bigoplus_{i=1}^N H^0(X, \mathcal{O}(km(D+A)) \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^{km\epsilon}),$$

$W_{m,k}^\circ$  maps isomorphically onto

$$\bigoplus_{i=1}^N H^0(X, \mathcal{O}(km(D+A) - m_1 a B_0) \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^{km\epsilon})$$

and  $W_{m,k}$  is contained in the image of

$$\mathrm{Sym}^k W_{m,1} \rightarrow H^0(X, \mathcal{O}(km(D+A))),$$

while  $W_{m,k}^\circ$  is contained in the image of

$$W_{m,1}^\circ \otimes \mathrm{Sym}^{k-1} W_{m,1} \rightarrow H^0(X, \mathcal{O}(km(D+A))).$$

Note that  $W_{m,k}^\circ$  is the analogue of the image of the map  $\alpha_{m,k}$  in the top diagram on p.26.

By construction, it follows that  $W_{m,k} \subseteq H^0(X, \mathcal{O}(km(D+A)) \otimes \mathfrak{b}_{|m(D+A)|}^{(k)})$ . On the other hand, since  $V$  is not contained in  $\mathbf{B}(D+A)$  and  $m$  is divisible enough (recall also that the points  $x_i$  are very general on  $V$ ), there are sections  $t_{m,k} \in H^0(X, \mathcal{O}(kma(D+A)))$  that do not vanish at any of the points  $x_i$ . Multiplying by the section  $t_{m,k}$  induces an embedding of  $W_{m,k}$  in  $H^0(X, \mathcal{O}(km(a+1)(D+A)) \otimes \mathfrak{b}_{|m(D+A)|}^k)$ . If we denote by  $\widetilde{W}_{m,k}$  its image, then it is clear that it satisfies the claimed properties. We deduce as before that if we replace  $B_0$  by a very general element  $B'_0 \in |m_1 a B|$  passing through  $x_1, \dots, x_N$ , then it is still true that every nonzero restriction to  $B'_0$  of an element in  $\widetilde{W}_{m,k}$  has order  $< km\epsilon$  at some  $x_i$ .

In particular, the  $\widetilde{W}_{m,k}$  also satisfy the lower bound (27). We apply Lemma 5.13 and Remark 5.14 for  $L = (D+A)$ , with  $m(a+1)$  instead of  $m$ , to get as before spaces of sections

$$W'_{m,k} \subseteq H^0(V_{m,k}, \mathcal{O}(km(a+1)(D+A-aB)) \otimes \mathfrak{b}_{|m(D+A)|}^{(k)})$$

such that

$$\dim W'_{m,k} \geq \dim \widetilde{W}_{m,k} - a(a+1)^d \cdot \deg_B(V) \cdot \ell(\mathcal{O}_{V_{m,k},\eta}) \cdot (km)^d.$$

On the other hand, the lower bound (31) still holds, and combining this with (27) and (34) we deduce that if  $m$  is divisible enough, then  $\dim W'_{m,k}$  grows like a polynomial of degree  $n$  in  $k$  when  $k$  goes to infinity.

Now since  $\mathcal{I} \otimes \mathcal{O}(B_1)$  is globally generated, the support of  $\mathcal{I}$  does not contain  $V$ , and  $V_{m,k}$  is defined by a primary ideal, we get an embedding

$$(36) \quad W'_{m,k} \hookrightarrow H^0(V_{m,k}, \mathcal{O}(km(a+1)(D+A-aB_2)) \otimes \mathcal{I}^{kma(a+1)} \mathfrak{b}_{|m(D+A)|}^{(k)}).$$

Since  $\mathcal{I}^{kma(a+1)} \subseteq \mathcal{I}^{kma}$ , applying Lemma 5.15 for  $p = ma$  and  $\gamma = 1/a$ , we deduce

$$(37) \quad W'_{m,k} \hookrightarrow H^0(V_{m,k}, \mathcal{O}(km(a+1)(D+A-aB_2)) \otimes \mathfrak{b}_{|(a+1)(D+A-aB_2)|}^{(k)_2}).$$

Furthermore, since  $aB_2 - A - H$  is ample, we get an embedding

$$(38) \quad W'_{m,k} \hookrightarrow H^0 \left( V_{m,k}, \mathcal{O}(km(a+1)(D-H)) \otimes \mathfrak{b}_{|m(a+1)(D-H)|}^{(k)_2} \right).$$

Note that we may assume that (36) and (38) are induced by multiplication with sections that do not vanish at any of the  $x_i$ , so they do not increase the order of vanishing at these points.

Because all sections in  $W'_{m,k}$  vanish on the subscheme defined by  $\mathfrak{b}_{|m(D-H)|}^{(k)_2}$ , it follows that they can be extended by zero to give a space  $W''_{m,k}$  of sections of  $\mathcal{O}(km(a+1)(D-H))$  on the subscheme defined by  $\mathfrak{b}_{|m(D-H)|}^{(k)}$ . As before, we show using a resolution  $\pi_m : X_m \rightarrow X$  that for  $k \gg 0$ , most of the sections in  $W''_{m,k}$  extend to  $X$ . On the other hand, for every nonzero section  $s$  in  $W''_{m,k}$  we can find  $i$  such that  $\text{ord}_{x_i}(s) \leq km\epsilon - 1$ . Since by our choice of  $\beta$  we have

$$\text{ord}_V(|km(a+1)(D-H)|) \geq \beta km(a+1) \|H\| > km\epsilon(a+1),$$

this gives a contradiction and completes the proof of the Theorem in the general case.  $\square$

*Proof of Lemma 5.15.* As  $\mathfrak{b}_{|p\gamma(D+A)|}^{(k)} = \mathfrak{b}_{|p\gamma(D+A)|}^{(k)_2}$ , it follows from (32) that it is enough to have

$$\mathcal{I}^{kp} \subseteq \mathfrak{b}_{|p(D+A-\frac{\gamma+1}{\gamma}B_2)|}^{(k)_2}.$$

Moreover, if we have this inclusion for  $k = 1$ , then we get it for all  $k$ . Note that  $(D + A - \frac{\gamma+1}{\gamma}B_2) - (D - \overline{B})$  is ample, so that  $\mathfrak{b}_{|p(D-\overline{B})|}^{(1)_2} \subseteq \mathfrak{b}_{|p(D+A-\frac{\gamma+1}{\gamma}B_2)|}^{(1)_2}$ . Therefore it is enough to choose  $q$  such that  $q(D - \overline{B})$  is integral and the reduced base locus of  $|q(D - \overline{B})|$  is  $\mathbf{B}_+(D)$ , and to take  $\mathcal{I} = \mathfrak{b}_{|q(D-\overline{B})|}^{(1)_2}$ .  $\square$

## 6. MOVING SESHADRI CONSTANTS

Moving Seshadri constants have been introduced in [Na2] for the description of the augmented base locus. In the case of nef line bundles, they coincide with the usual Seshadri constants. In this section we prove the basic properties of these invariants, and we use the results in the previous sections to deduce a stronger version of the main result in [Na2]. If  $D$  is a nef  $\mathbf{Q}$ -divisor on a smooth, projective variety  $X$ , we denote by  $\epsilon(D; x)$  the Seshadri constant of  $D$  at  $x$ . For the definition and basic results on Seshadri constants we refer to [Laz] §5.1.

As in the case of asymptotic intersection numbers, there are two equivalent definitions for moving Seshadri constants. We start this time with the definition in terms of arbitrary decompositions for the pull-back of  $D$ , a definition which applies to arbitrary  $\mathbf{R}$ -divisors (note the similarity with the formula in Proposition 2.11). Suppose that  $x \in X$  and that  $D$  is a divisor such that  $x \notin \mathbf{B}_+(D)$ . We consider projective morphisms  $f : X' \rightarrow X$ , with  $X'$  smooth, which are isomorphisms over a neighborhood of  $x$ , and decompositions  $f^*(D) = A + E$ , with  $A$  an ample  $\mathbf{Q}$ -divisor and  $E$  effective such that  $f^{-1}(x)$

is not in the support of  $E$ . Note that for every such  $f$ , we have  $f^{-1}(x) \notin \mathbf{B}_+(f^*(D))$ , so there exist indeed decompositions as described above.

**Definition 6.1.** Let  $D$  be an  $\mathbf{R}$ -divisor. If  $x \notin \mathbf{B}_+(D)$ , then the *moving Seshadri constant* of  $D$  at  $x$  is

$$(39) \quad \epsilon(\| D \|; x) := \sup_{f^*(D)=A+E} \epsilon(A, x),$$

where the supremum is over all morphisms  $f$  and decompositions  $f^*(D) = A + E$  as above. If  $x \in \mathbf{B}_+(D)$ , then we put  $\epsilon(\| D \|; x) = 0$ .

It is easy to see that the above invariant is finite (see, for example, Proposition 6.3 i) below). It is also clear from the definition that  $x$  is in  $\mathbf{B}_+(D)$  if and only if  $\epsilon(\| D \|; x) = 0$ . The value 0 over  $\mathbf{B}_+$  is justified by the following theorem, which is our main result on moving Seshadri constants. As we will see, it can be considered a stronger version of Theorem 0.8 in [Na2].

**Theorem 6.2.** *For every point  $x$  in  $X$ , the map  $D \longrightarrow \epsilon(\| D \|; x)$  is continuous on the entire Néron-Severi space  $N^1(X)_{\mathbf{R}}$ .*

The proof will be given at the end of this section. We start by giving some basic properties and interpretations of the moving Seshadri constants. As moving Seshadri constants of non-big divisors are trivial, we henceforth assume that all divisors are big.

**Proposition 6.3.** *Suppose that  $D$  is a big  $\mathbf{R}$ -divisor on  $X$ .*

- i) *We have  $\epsilon(\| D \|; x) \leq \text{vol}_X(D)^{1/\dim(X)}$ .*
- ii) *If  $D \equiv E$ , then  $\epsilon(\| D \|; x) = \epsilon(\| E \|; x)$ .*
- iii)  *$\epsilon(\| \lambda D \|; x) = \lambda \cdot \epsilon(\| D \|; x)$  for every positive  $\lambda$ .*
- iv) *If  $D$  is an ample  $\mathbf{Q}$ -divisor, then  $\epsilon(\| D \|; x) = \epsilon(D; x)$ .*
- v) *If  $D'$  is another  $\mathbf{R}$ -divisor such that  $x \notin \mathbf{B}_+(D) \cup \mathbf{B}_+(D')$ , then*

$$(40) \quad \epsilon(\| D + D' \|; x) \geq \epsilon(\| D \|; x) + \epsilon(\| D' \|; x).$$

*Proof.* All proofs follow from definition and from the properties of the usual Seshadri constants.  $\square$

We explain now the connection with the definition of moving Seshadri constants from [Na2]. This is analogous to the definition of asymptotic intersection numbers. Suppose that  $D$  is a  $\mathbf{Q}$ -divisor and that  $x \notin \mathbf{B}(D)$ . Let  $m$  be sufficiently divisible, so  $mD$  is an integral divisor and  $x$  is not in the base locus of  $|mD|$ . We take a resolution of this base locus as in Definition 2.6 (with  $V$  replaced by  $x$ ).

We define following [Na2]:

$$(41) \quad \epsilon'(\| D \|; x) := \lim_{m \rightarrow \infty} \frac{\epsilon(M_m; \pi_m^{-1}(x))}{m} = \sup_m \frac{\epsilon(M_m; \pi_m^{-1}(x))}{m},$$



where the limit and the supremum are over divisible enough  $m$ . Note that  $\epsilon(M_m; x)$  does not depend on the particular morphism  $\pi_m$ . Moreover, given positive integers  $p$  and  $q$ , sufficiently divisible, we may choose  $\pi : X' \rightarrow X$  that satisfies our requirements for  $|pD|$ ,  $|qD|$  and  $|(p+q)D|$ . Since we have  $M_{p+q} = M_p + M_q + E$ , for some effective divisor  $E$  with  $\pi^{-1}(x) \notin \text{Supp}(E)$ , we deduce that

$$\epsilon(M_{p+q}; \pi^{-1}(x)) \geq \epsilon(M_p; \pi^{-1}(x)) + \epsilon(M_q; \pi^{-1}(x)).$$

This implies that the limit in the definition of  $\epsilon'(\|D\|; x)$  exists, and it is equal to the corresponding supremum. We now show that the two invariants we have defined are the same.

**Proposition 6.4.** *If  $D$  is a big  $\mathbf{Q}$ -divisor and if  $x \notin \mathbf{B}(D)$ , then  $\epsilon'(\|D\|; x) = \epsilon(\|D\|; x)$ .*

*Proof.* By replacing  $D$  with a suitable multiple, we may assume that  $D$  is integral,  $\mathbf{B}(D) = \text{Bs}(|D|)_{\text{red}}$  and that  $|D|$  defines a rational map whose image has dimension  $n$ . For  $m \in \mathbf{N}^*$ , take  $\pi_m : X_m \rightarrow X$  as in the definition of  $\epsilon'(\|D\|; x)$  and write  $\pi_m^*(mD) = M_m + E_m$ . Recall our assumption that  $\pi_m^{-1}(x) \notin \text{Supp}(E_m)$ .

Suppose first that  $x \in \mathbf{B}_+(D)$ . In this case, it follows easily that  $\pi_m^{-1}(x) \in \mathbf{B}_+(M_m)$ . Since  $M_m$  is big and nef and  $\pi_m^{-1}(x) \in \mathbf{B}_+(M_m)$ , Corollary 5.6 implies that there is a subvariety  $V \subseteq X_m$  of dimension  $d \geq 1$ , such that  $\pi_m^{-1}(x) \in V$  and  $(M_m^d \cdot V) = 0$ . Therefore  $\epsilon(M_m; \pi_m^{-1}(x)) = 0$ , and since this is true for every  $m$  we get  $\epsilon'(\|D\|; x) = 0$ .

Suppose now that  $x \notin \mathbf{B}_+(D)$ . We show first that  $\epsilon(\|D\|; x) \geq \epsilon'(\|D\|; x)$  by proving that for every  $m$  divisible enough,  $\epsilon(\|D\|; x) \geq \epsilon(M_m; \pi_m^{-1}(x))/m$ . If  $\pi_m^{-1}(x) \in \mathbf{B}_+(M_m)$ , then the above argument using Corollary 5.6 shows that  $\epsilon(M_m; \pi_m^{-1}(x)) = 0$ , and we are done.

Therefore we may assume that  $\pi_m^{-1}(x) \notin \mathbf{B}_+(M_m)$ , so we can write  $M_m = A + E$ , where  $A$  is ample,  $E$  is effective, and  $\pi_m^{-1}(x) \notin \text{Supp}(E)$ . If  $p \in \mathbf{N}^*$ , then we have  $M_m = (1/p)E + A_p$ , where  $A_p = \frac{1}{p}A + \frac{p-1}{p}M_m$  is ample. It follows from definition that  $\epsilon(\|D\|; x) \geq (1/m)\epsilon(A_p; \pi_m^{-1}(x))$  for every  $p$ . By letting  $p$  go to infinity, we deduce  $\epsilon(\|D\|; x) \geq \epsilon(M_m; \pi_m^{-1}(x))/m$ .

We prove now that  $\epsilon(\|D\|; x) \leq \epsilon'(\|D\|; x)$ . Let  $f : X' \rightarrow X$  and  $f^*(D) = A + E$  be as in the definition of  $\epsilon(\|D\|; x)$ . Fix  $m$  such that  $mA$  is integral and very ample. By taking a log resolution of the base locus of  $f^*(mD)$  which is an isomorphism over a neighborhood of  $f^{-1}(x)$ , we may assume that we can write  $f^*(mD) = M_m + E_m$  as in the definition of  $\epsilon'(\|D\|; x)$ . Since  $mA$  is basepoint-free, we have  $M_m = mA + E'_m$ , where  $E'_m$  is effective and  $E'_m \leq mE$ , so  $f^{-1}(x) \notin \text{Supp}(E'_m)$ . Therefore  $\epsilon(M_m; f^{-1}(x))/m \geq \epsilon(A; f^{-1}(x))$ , hence  $\epsilon'(\|D\|; x) \geq \epsilon(\|D\|; x)$ , and this completes the proof of the Proposition.  $\square$

**Remark 6.5.** If  $D$  is a  $\mathbf{Q}$ -divisor that is nef and big, then  $\epsilon(\|D\|; x) = \epsilon(D; x)$ . This follows of course from the corresponding property for ample divisors, together with the continuity property of both invariants (see Theorem 6.2 above). However, we can also give a direct argument as follows. If  $x \in \mathbf{B}_+(D)$ , then  $\epsilon(\|D\|; x) = 0$  by definition, while  $\epsilon(D; x) = 0$  by Corollary 5.6. Suppose now that  $x \notin \mathbf{B}_+(D)$ . If  $f : X' \rightarrow X$  and

$f^*(D) = A + E$  are as in the definition of  $\epsilon(\|D\|; x)$ , using the fact that  $x$  is not in  $\text{Supp}(E)$  we deduce

$$\epsilon(D; x) = \epsilon(f^*(D); f^{-1}(x)) \geq \epsilon(A; f^{-1}(x)).$$

This gives  $\epsilon(D, x) \geq \epsilon(\|D\|, x)$ . On the other hand, since  $D$  is nef, the argument in the proof of Proposition 6.4 shows that we can write  $D = A_p + \frac{1}{p}E$ , with  $A_p$  ample and  $E$  effective, with  $x \notin \text{Supp}(E)$ . By definition, we have  $\epsilon(\|D\|; x) \geq \epsilon(A_p; x)$  for all  $p$ , and letting  $p$  go to infinity we get  $\epsilon(\|D\|; x) \geq \epsilon(D; x)$ .

The moving Seshadri constants measure asymptotic separation of jets, as is the case of the usual constants (see [Laz], Theorem 5.1.17). We give now this interpretation. Recall that if  $L$  is a line bundle on a smooth variety  $X$ , we say that  $L$  separates  $s$ -jets at  $x \in X$  if the canonical morphism

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x^{s+1})$$

is surjective. Let  $s(L; x)$  be the smallest  $s \geq 0$  such that  $L$  separates  $s$ -jets at  $x$  (if there is no such  $s \geq 0$ , then we put  $s(L; x) = 0$ ).

**Proposition 6.6.** *If  $L$  is a big line bundle on  $X$ , then*

$$\epsilon(\|L\|; x) = \sup_{m \rightarrow \infty} \frac{s(mL; x)}{m} = \limsup_{m \rightarrow \infty} \frac{s(mL; x)}{m}.$$

*Proof.* We may clearly assume that  $x \notin \mathbf{B}(L)$ , the statement being trivial otherwise. Let  $m$  be such that  $x \notin \text{Bs}(|mL|)$ , so we have  $\pi_m : X_m \rightarrow X$  and  $\pi_m^*(mL) = M_m + E_m$ , as in the definition of  $\epsilon'(\|L\|; x)$ . Since  $\pi_m$  is an isomorphism over a neighborhood of  $x$ , it induces an isomorphism

$$mL \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x^{s+1} \simeq \pi_m^*(mL) \otimes \mathcal{O}_{X_m,x'}/\mathfrak{m}_{x'}^{s+1},$$

where  $x' = \pi_m^{-1}(x)$ . As  $\pi_m$  also induces an isomorphism  $H^0(X, mL) \simeq H^0(X_m, \pi_m^*(mL))$ , we deduce  $s(mL; x) = s(\pi_m^*(mL); x')$ .

On the other hand, as  $x' \notin \text{Supp}(E_m)$ , multiplication by a local equation of  $E_m$  induces an isomorphism

$$M_m \otimes \mathcal{O}_{X_m,x'}/\mathfrak{m}_{x'}^{s+1} \simeq \pi_m^*(mL) \otimes \mathcal{O}_{X_m,x'}/\mathfrak{m}_{x'}^{s+1}.$$

Moreover, since  $E_m$  is the fixed part of  $\pi_m^*(mL)$ , we have an isomorphism  $H^0(X_m, M_m) \simeq H^0(X_m, \pi_m^*(mL))$ . This gives  $s(\pi_m^*(mL); x') = s(M_m; x')$ .

We show first that  $\epsilon(\|L\|; x) \geq s(mL; x)/m$  for every  $m$ . Since  $s(pmL; x) \geq p \cdot s(mL; x)$  for every  $p$ , we may assume that  $m$  is divisible enough, so  $x \notin \text{Bs}(|D|)$ . We take  $\pi_m$  and a decomposition as above. The fact that  $\epsilon(M_m; x') \geq s(M_m; x')$  follows as in [Laz], *loc. cit.*: let  $C$  be an integral curve passing through  $x'$  and suppose that  $M_m$  separates  $s$ -jets at  $x'$ . We can find  $F \in |M_m|$  such that  $\text{mult}_{x'}(F) \geq s$  and  $C \not\subseteq F$ . This gives  $(F \cdot C) \geq s \cdot \text{mult}_x C$ , hence  $\epsilon(M_m; x') \geq s$ . Since  $\epsilon(\|L\|; x) \geq \epsilon(M_m; x')/m$  by Proposition 6.4 and since  $s(M_m; x') = s(mL; x)$ , we deduce  $\epsilon(\|L\|; x) \geq s(mL; x)/m$ .

In order to finish, it is enough to see that for every  $\eta > 0$ , we have  $s(mL; x)/m > \epsilon(\|L\|; x) - \eta$  for some  $m$ . If  $x \in \mathbf{B}_+(L)$ , then the assertion follows trivially. If  $x \notin \mathbf{B}_+(L)$ ,

then by definition we can find  $f : X' \rightarrow X$  which is an isomorphism over a neighborhood of  $x$ , with  $X'$  smooth, and a decomposition  $f^*(L) = A + F$ , where  $A$  and  $F$  are  $\mathbf{Q}$ -divisors, with  $A$  ample,  $F$  effective,  $x' = f^{-1}(x) \notin \text{Supp}(F)$  and  $\epsilon(A; x') \geq \epsilon(\|L\|; x) - \eta/2$ . Since  $A$  is ample, it follows from [Laz], *loc. cit.*, that we can find  $m$  such that  $mA$  is an integral divisor and  $s(mA; x')/m \geq \epsilon(A; x') - \eta/2$ . Therefore it is enough to show that  $s(mL; x) \geq s(mA; x')$ . This follows by the same arguments as before, as  $\pi$  being an isomorphism over a neighborhood of  $x$  and  $x' \notin \text{Supp}(E)$  imply  $s(mL; x) = s(mf^*(L); x') \geq s(mA; x')$ .  $\square$

We use Theorems 2.13 and 5.7 to extend the relation between Seshadri constants and volumes to the case of big line bundles.

**Proposition 6.7.** *If  $D$  is a big  $\mathbf{Q}$ -divisor on  $X$  and if  $x \in X$ , then*

$$\epsilon(\|D\|; x) = \inf_{x \in V} \frac{\text{vol}_{X|V}(D)^{1/\dim(V)}}{\text{mult}_x V},$$

where the infimum is over all positive dimensional subvarieties  $V$  containing  $x$ .

*Proof.* If  $x \in \mathbf{B}_+(D)$ , then  $\epsilon(\|D\|; x) = 0$  and, on the other hand, by Theorem 5.7  $\text{vol}_{X|V}(D) = 0$  for any irreducible component  $V$  of  $\mathbf{B}_+(D)$  passing through  $x$ .

If  $x \notin \mathbf{B}_+(D)$ , then any  $V$  through  $x$  is not contained in  $\mathbf{B}_+(D)$ , and so we can apply Theorem 2.13. Thus we only need to prove that

$$(42) \quad \epsilon(\|D\|; x) = \inf_{x \in V} \frac{\|D^d \cdot V\|^{1/d}}{\text{mult}_x V},$$

where  $d = \dim(V)$ .

This however is immediate. Indeed, for each  $m$  divisible enough let  $\pi_m : X_m \rightarrow X$  and  $\pi_m^*(mD) = M_m + E_m$  be as in the definition of  $\epsilon'(\|D\|; x)$ , and for every  $V$  denote by  $\tilde{V}_m \subseteq X_m$  the proper transform of  $V$ . Since  $M_m$  is nef, we have

$$\inf_{x \in V} \frac{(M_m^d \cdot \tilde{V}_m)^{1/d}}{\text{mult}_x V} = \epsilon(M_m; \pi_m^{-1}(x)).$$

It is straightforward to deduce now equation (42) from Proposition 6.4 and the definition of  $\|D^d \cdot V\|$ .  $\square$

The proof of the continuity of the moving Seshadri constants is now a formal consequence of the above results.

*Proof of Theorem 6.2.* Let  $D$  be a  $\mathbf{Q}$ -divisor such that  $x \notin \mathbf{B}_+(D)$ . Then the formal concavity property of the moving Seshadri constant (Proposition 6.3 v) gives, precisely as in the proof of Theorem 5.2 on restricted volumes, that our function is locally uniformly continuous around  $D$ . We do not repeat the argument here. In order to finish, it is enough to show that if  $D$  is a real class such that  $x$  is in  $\mathbf{B}_+(D)$ , then  $\lim_{D' \rightarrow D} \epsilon(\|D'\|; x) = 0$ . It is clear that it is enough to consider only those  $D'$  that are big. If  $V$  is an irreducible component of  $\mathbf{B}_+(D)$ , then Theorem 5.2 gives  $\lim_{D' \rightarrow D} \text{vol}_{X|V}(D') = 0$ , so we conclude by Proposition 6.7. (This part of the theorem is a strengthening of the main result in [Na2] to the case of arbitrary real divisor classes.)  $\square$

Finally, let's observe that the moving Seshadri constant at a point controls the separation properties of the corresponding "adjoint" linear series at that point, as in the case of ample line bundles and usual Seshadri constants (cf. [Dem] or [EKL] – the proof is essentially the same, with a slight variation due to the initial non-positivity):

**Proposition 6.8.** *Let  $L$  be a big line bundle and assume that for some  $x \in X$*

$$\epsilon(\|L\|; x) > \frac{s+n}{p},$$

*where  $s \geq 0$  and  $p > 0$  are integers, and  $n = \dim(X)$ . Then the linear series  $|K_X + pL|$  separates  $s$ -jets at  $x$ .*

*Proof.* Note that the hypothesis implies that  $x \notin \mathbf{B}_+(L)$ . Let  $m$  be divisible and large enough, and let  $\pi_m: X_m \rightarrow X$  be as in the definition of  $\epsilon'(D; x)$ . If we write  $\pi_m^*(mL) = M_m + E_m$ , then we have  $\epsilon(M_m; x) > \frac{m(s+n)}{p}$ . For simplicity, we identify  $x$  with its inverse image in  $X_m$ . We need to prove the surjectivity of the restriction map

$$H^0(X, \mathcal{O}_X(K_X + pL)) \longrightarrow H^0(X, \mathcal{O}_X(K_X + pL) \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1}).$$

Since  $K_{X_m/X}$  is supported on the exceptional locus (so, in particular,  $x$  does not lie in its support), this is equivalent to the surjectivity of the map

$$H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + \pi_m^*(pL))) \longrightarrow H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + \pi_m^*(pL)) \otimes \mathcal{O}_{X_m}/\mathfrak{m}_x^{s+1}).$$

This in turn is implied by the surjectivity of the restriction map

$$(43) \quad H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + \lceil \frac{p}{m} M_m \rceil)) \longrightarrow H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + \lceil \frac{p}{m} M_m \rceil) \otimes \mathcal{O}_{X_m}/\mathfrak{m}_x^{s+1}),$$

since the sections on the left hand side inject into  $H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + \pi_m^*(pL)))$  by twisting with the equation of the effective integral divisor  $\lfloor \frac{p}{m} E_m \rfloor$ , and  $x$  is not in the support of this divisor. Since  $M_m$  is big and nef, the argument for (43) goes as usual: consider  $f: X'_m \rightarrow X_m$  the blow-up of  $X_m$  at  $x$ , with exceptional divisor  $E$ . It is enough to prove the vanishing of  $H^1(X_m, \mathcal{O}_{X_m}(K_{X_m} + \lceil \frac{p}{m} M_m \rceil) \otimes \mathfrak{m}_x^{s+1})$ , which in turn holds if

$$H^1(X'_m, f^* \mathcal{O}_{X_m}(K_{X_m} + \lceil \frac{p}{m} M_m \rceil) \otimes \mathcal{O}_{X'_m}(-(s+1)E)) = 0.$$

We can rewrite this last divisor as  $K_{X'_m} + f^* \lceil \frac{p}{m} M_m \rceil - (s+n)E$ , and the required vanishing is a consequence of the Kawamata-Viehweg Vanishing Theorem: using the lower bound on the Seshadri constant of  $M_m$  at  $x$ , we see that  $f^* \lceil \frac{p}{m} M_m \rceil - (s+n)E$  is the round-up of an ample  $\mathbf{Q}$ -divisor. (Note that we are implicitly using here that since  $M_m$  is globally generated, the general divisor in the corresponding linear series will avoid  $x$ , and so we can arrange that  $f^* \lceil \frac{p}{m} M_m \rceil = \lceil \frac{p}{m} f^* M_m \rceil$ .)  $\square$

**Remark 6.9.** The first step in the above proof consisted in reducing to the case when  $L$  is big and nef. The rest of the proof could be alternatively recast in the language of multiplier ideals, as follows (see Chapter 9 in [Laz] for basic facts on multiplier ideals). Suppose that  $L$  is big and nef, and fix a rational number  $t$  with  $\frac{n+s}{p} < t < \epsilon(L; x)$ . Proposition 6.6 implies that there is  $m > 0$  such that  $mt$  is an integer and  $mL$  separates  $mt$  jets at  $x$ . In particular, we can find  $D \in |mL|$  whose tangent cone at  $x$  is the cone over a smooth hypersurface of degree  $mt$ . This implies that in a neighborhood of  $x$  we

have  $\mathcal{J}(qD) = \mathfrak{m}_x^{s+1}$ , where  $q = \frac{n+s}{mt}$ . Since  $(p - qm)L$  is big and nef, Nadel's Vanishing Theorem implies that  $H^1(X, \mathcal{O}(K_X + pL) \otimes \mathcal{J}(qD)) = 0$ , hence the map

$$H^0(X, \mathcal{O}(K_X + pL)) \rightarrow H^0(Z, \mathcal{O}(K_X + pL)|_Z)$$

is surjective, where  $Z$  is the subscheme defined by  $\mathcal{J}(qD)$ . We now deduce the assertion in the proposition using the fact that  $Z$  is defined by  $\mathfrak{m}_x^{s+1}$  in a neighborhood of  $x$ .

**Acknowledgements.** We are grateful to Olivier Debarre and Mihai Păun for useful discussions. We would like to thank also the anonymous referee for several remarks and suggestions.

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